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A Characterization of the Real Zeros of a Particular Transcendental Function

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INTRODUCTION

Literature discussing the characterization of the real zeros of transcendental functions is conspicuously absent (ref. 1). As a result, scientists and engineers who wish to determine the zeros of such functions are at a severe disadvantage unless they have some prior knowledge concerning the location of the zeros. All the iterative schemes available require at least one estimate in order to initiate the algorithm. If the estimate is not sufficiently close to a real zero, or if no real zero exists, the iteration may diverge or lead to the "wrong" zero (ref. 2).

This recurrent problem is the motivation for this paper, which characterizes the real zeros of the transcendental function $y = ax + be^{cx}$ (and equivalent forms) where a, b, and c are real numbers and $e = 2.71828$.

This transcendental function was chosen because it is the solution of many first-order differential equations and sometimes appears in the numerical solution of nonlinear differential equations. Thus, this paper should facilitate the solution of many everyday problems, as well as have heuristic value.

The following discussion addresses the above problem with respect to this particular transcendental function by way of theorems. The theorems speak to the questions of the existence, bounds, and number of real zeros. The answers to these questions are particularly important in view of the value of computer resources, since they remove the inefficiency involved in starting an iteration from a poor initial estimate or in pursuing solutions that do not exist. It is hoped that this discussion will afford insight into other types of transcendental functions as well.

DISCUSSION

Proposition 1: The transcendental function $y = ax + be^{cx}$ has at most two distinct real zeros.

Proof: Let x_1 and x_2 be two zeros of y and let $x_1 < x_2$. Then, by Rolle's theorem, there exists a value x_t such that $x_1 < x_t < x_2$, where $y'(x_t) = a + cbe^{cx_t} = 0$ and y' denotes the derivative of y. Now suppose that there are more than two zeros of y. They may be ordered so that $x_1 < x_2 < x_3, \ldots, x_n$. From Rolle's theorem, there exists x_t such that $x_1 < x_t < x_2$, $x_2 < x_m < x_3$, and so on, where $x_t \neq x_m$ and $y'(x_t), y'(x_m) = 0$. Clearly, then, $a + cbe^{cx_t} = a + cbe^{cx_m} = 0$, which implies that $x_t = x_m$.

Theorem 1: Let $y = ax + be^{cx}$ and let x_0 be a real zero for y. Then, if $cx_0 > 0$, $\left|\frac{b}{a}\right| < |x_0| < \frac{2\left(\ln \left|\frac{2a}{bc}\right|\right)}{|c|}$. If $cx_0 < 0$, then $\frac{\ln 2}{\left|\frac{2a}{b}\right| + |c|} < |x_0| < \left|\frac{b}{a}\right|$.

Proof: Let $c > 0$ and $x_0 > 0$. Then $x_0 = -\frac{b}{a}e^{cx_0}$ and $\frac{c}{2}x_0 = -\frac{bc}{2a}e^{cx_0}$. Since $\frac{c}{2}x_0 > 0$, it follows that $-\frac{bc}{a} > 0$. Thus, $e^{\frac{c}{2}x_0} > -\frac{bc}{2a}e^{cx_0}$ and $-\frac{2a}{bc} > e^{\frac{cx_0}{2}}$, or $\ln -\frac{2a}{bc} > \frac{cx_0}{2}$, or $\frac{2\left(\ln \left|\frac{2a}{bc}\right|\right)}{|c|} > |x_0|$. From the original equation, $|x_0| = \left|\frac{b}{a}\right|e^{cx_0} > \left|\frac{b}{a}\right|$; that is, $\left|\frac{b}{a}\right| < |x_0| < \frac{2 \ln \left|\frac{2a}{bc}\right|}{|c|}$.

Consider now the case in which $c > 0$ and $x_0 < 0$ such that $cx_0 < 0$. Then we may write $-ax_0 + be^{-cx_0} = 0$ where in this form $x_0 > 0$ and $c > 0$. Thus $x_0 = \frac{b}{a}e^{-cx_0}$ where $\frac{b}{a} > 0$ and $|x_0| = \left|\frac{b}{a}\right|e^{-cx_0} < \left|\frac{b}{a}\right|$. Further, $x_0 = \frac{b}{a}e^{-cx_0}$ implies that $|x_0| = \left|\frac{b}{a}\right|e^{-cx_0}$ and that $2\left|\frac{a}{b}\right||x_0| = 2e^{-cx_0}$. Clearly, $e^{2\left|\frac{a}{b}\right||x_0|} > 2e^{-|cx_0|}$ or $e^{2\left|\frac{a}{b}\right||x_0|+|cx_0|} > 2$ and $2\left|\frac{a}{b}\right||x_0| + |cx_0| > \ln 2$. Simplifying, $|x_0| > \frac{\ln 2}{\left|\frac{2a}{b}\right| + |c|}$. That is, $\frac{\ln 2}{\left|\frac{2a}{b}\right| + |c|} < |x_0| < \left|\frac{b}{a}\right|$.

Now let $c < 0$ and $x_0 > 0$. Then we may write $ax_0 + be^{-cx_0} = 0$ where $x_0 > 0$ and $c > 0$ in this form. Thus, $x_0 = -\frac{b}{a}e^{-cx_0}$ where $-\frac{b}{a} > 0$ and $|x_0| = \left|\frac{b}{a}\right| \frac{1}{e^{cx_t}} < \left|\frac{b}{a}\right|$. For the same reasons as in the case where $x_0 < 0$ and $c > 0$, it follows that $|x_0| > \frac{\ln 2}{\left|\frac{2a}{b}\right| + |c|}$.

In the final case, $c < 0$, and $x_0 < 0$. Again, we may write $-ax_0 + be^{cx_0} = 0$ where $x_0 > 0$ and $c > 0$ in this form. Thus, $x_0 = \frac{b}{a}e^{cx_0}$

where $\frac{b}{a} > 0$. As in the case where $c > 0$ and $x_0 > 0$, it follows that $\left|\frac{b}{a}\right| < |x_0| < \frac{2 \ln \left|\frac{2a}{bc}\right|}{|c|}$.

Theorem 2: If the function $y = ax + be^{cx}$ has two real zeros, both zeros have the sign of c. Further, where $|x_2| > |x_1|$, $\left|\frac{b}{a}\right| < |x_1| < \frac{1}{|c|}$ and $\frac{1}{|c|} < |x_2| < \frac{2 \ln \left|\frac{2a}{bc}\right|}{|c|}$.

Proof: Let $c > 0$ and let x_1 and x_2 be two zeros where $x_1 < x_2$. Then $ax_2 + be^{cx_2} = 0$ and $ax_1 + be^{cx_1} = 0$. From Rolle's theorem, there exist x_t such that $x_1 < x_t < x_2$ and $y'(x_t) = 0$; that is, $a + bce^{cx_t} = 0$, or $a = -bce^{cx_t}$. Substituting in $ax_2 + be^{cx_2} = 0$ yields $\left(-cbe^{cx_t}\right)x_2 + be^{cx_2} = 0$, or $x_2 = \frac{e^{c(x_2-x_t)}}{c}$, which implies that $x_2 > 0$. Similarly, $x_1 = \frac{e^{c(x_1-x_t)}}{c}$ and $x_1 > 0$. Thus, x_2 and x_1 have the sign of c, and hence $cx_1 > 0$ and $cx_2 > 0$. From the expressions for x_1 and x_2 and the fact that $x_1 < x_t < x_2$, it follows that $|x_1| < \frac{1}{|c|}$ and that $|x_2| > \frac{1}{|c|}$. From theorem 1, it follows that $\left|\frac{b}{a}\right| < |x_1| < \frac{1}{|c|}$ and that $\frac{1}{|c|} < |x_2| < \frac{2 \ln \left|\frac{2a}{bc}\right|}{|c|}$. Further, $x_1 < x_2$ implies that $|x_2| > |x_1|$.

In the case where $c < 0$, the same argument leads to the following expressions:

$$x_1 = \frac{e^{c(x_1-x_t)}}{c}$$

and

$$x_2 = \frac{e^{c(x_2-x_t)}}{c}$$

where $x_1 < x_t < x_2$. We may rewrite the expressions as follows:

$$x_1 = \frac{e^{-c(x_1-x_t)}}{-c}$$

and

$$x_2 = \frac{e^{-c(x_2-x_t)}}{-c}$$

where $c > 0$. Thus, $x_1 = \frac{e^{c(x_t-x_1)}}{-c}$ and $x_2 = \frac{e^{c(x_t-x_2)}}{-c}$. These expressions indicate that $x_1 < 0$ and $x_2 < 0$, implying that $x_t < 0$. Thus,

$$|x_1| = \left|\frac{1}{c}\right| e^{c(x_t-x_1)} > \frac{1}{|c|}$$

and

$$|x_2| = \left|\frac{1}{c}\right| e^{c(x_t-x_2)} < \frac{1}{|c|}$$

From theorem 1,

$$\left|\frac{b}{a}\right| < |x_2| < \frac{1}{c} < |x_1| < \frac{\left|2 \ln \left|\frac{2a}{bc}\right|\right|}{|c|}$$

and the conditions that $x_1 < 0$ and $x_2 < 0$ imply that $|x_1| > |x_2|$.

Theorem 3: Let $abc < 0$. Then $-\frac{a}{bc} > e$ if and only if the function $y = ax + be^{cx}$ has two distinct zeros.

Proof: Let y have two distinct zeros, x_1 and x_2. Then, by theorem 2, one zero lies in the interval $\left(0, \frac{1}{c}\right)$. Also, $\left[y(0)\right]\left[y\left(\frac{1}{c}\right)\right] < 0$, because if this were not the case, x_1, a zero, would also be an extremum such that $y'(x_1) = 0$. But by Rolle's theorem there exists another point, x_t, such that $x_1 < x_t < x_2$ and that $y'(x_t) = 0$, which is impossible for this particular function. Hence $\left[y(0)\right]\left[y\left(\frac{1}{c}\right)\right] = b\left(\frac{a}{c} + be\right)$, which is less than zero. That is, $\frac{a}{bc} + e < 0$ or $-\frac{a}{bc} > e$.

Conversely, let $\frac{-a}{bc} > e$. Then $0 < \left|\frac{1}{c}\right| < \frac{\left|2 \ln \frac{-2a}{bc}\right|}{|c|}$. Observe that $\left[y(0)\right]\left[y\left(\frac{1}{c}\right)\right] = b\left(\frac{a}{c} + be\right)$. But the sign of $\frac{b\left(\frac{a}{c} + be\right)}{b^2}$ is that of $\left[y(0)\right]\left[y\left(\frac{1}{c}\right)\right]$.

That is, $\dfrac{b\left(\frac{a}{c} + be\right)}{b^2} = \dfrac{a}{bc} + e < 0$ by hypothesis. Thus, by the intermediate value theorem, there exists at least one zero in the interval $\left(0, \frac{1}{c}\right)$. In addition,

$$\left[y\left(\frac{1}{c}\right)\right]\left[y\left(\frac{2 \ln \frac{-2a}{bc}}{c}\right)\right] = \left(\frac{a}{b} + be\right)\left[\frac{2a}{c}\left(\ln \frac{-2a}{bc} + \frac{2a}{bc}\right)\right]$$

But the sign of this product depends on the sign of $\left(\frac{a}{c} + be\right)\left(\frac{2a}{c}\right)$ since $\ln \frac{-2a}{bc} + \frac{2a}{bc} < 0$. But $\left(\frac{a}{c} + be\right)\left(\frac{2a}{c}\right)$ has the same sign as $\left(\frac{a}{c} + be\right)\frac{c}{a}$, which equals $1 + \frac{bc}{a}e$. From this hypothesis, it follows that $1 + \frac{bc}{a}e > 0$ and that $\left[y\left(\frac{1}{c}\right)\right]\left[y\left(\frac{2 \ln \frac{-2a}{bc}}{c}\right)\right] < 0$. Thus, by the intermediate value theorem, there is a zero in the interval $\left(\frac{1}{c}, \frac{2 \ln \frac{-2a}{bc}}{c}\right)$. By proposition 1, there are exactly two.

Theorem 4: Let $abc < 0$. Then $\frac{-a}{bc} = e$ if and only if there is exactly one zero for $y = ax + be^{cx}$.

Proof: Let $\frac{-a}{bc} = e$. Then observe that $y\left(\frac{1}{c}\right) = \frac{a}{c} + be$. But the expression $be = \frac{-a}{c}$ implies that $y\left(\frac{1}{c}\right) = 0$. That is, $\frac{1}{c}$ is a zero for y, and from theorem 3 it is the only zero.

Conversely, let x_0 be the only zero for y. Then $ax_0 + be^{cx_0} = 0$ and $cx_0 = -\frac{bc}{a}e^{cx_0}$. But this says that $cx_0 > 0$, and from theorem 1, $|x_0| < \left|\frac{2 \ln \frac{-2a}{bc}}{c}\right|$. That is, x_0 belongs to the interval $\left(0, \frac{2 \ln \frac{-2a}{bc}}{c}\right)$. But $\left[y(0)\right]\left[y\left(\frac{2 \ln \frac{-2a}{bc}}{c}\right)\right] = \frac{2ab}{c}\left[\ln \frac{-2a}{bc} + \frac{2a}{bc}\right] > 0$. Thus, this implies that x_0 is an extremum, and hence $y'(x_0) = a + be^{cx_0} = 0$ and $x_0 = \dfrac{\ln \frac{-a}{bc}}{c}$. Upon substitution, we have

$$\frac{a \ln \frac{-a}{bc}}{c} + be^{\left(\ln \frac{-a}{bc}\right)} = \frac{a}{c} \ln \frac{-a}{bc} + b\left(\frac{-a}{bc}\right)$$

$$= \frac{a}{c}\left(\ln \frac{-a}{bc} - 1\right)$$

$$= 0$$

This implies that $\ln \frac{-a}{bc} = 1$, which implies that $\frac{-a}{bc} = e$.

Theorem 5: Let $abc < 0$. Then $-\frac{a}{bc} < e$ if and only if there are no zeros for $y = ax + be^{cx}$.

Proof: The proof of this theorem follows immediately from theorems 3 and 4.

Theorem 6: $abc > 0$ if and only if $y = ax + be^{cx}$ has exactly one zero, x_0, such that $cx_0 < 0$.

Proof: Let y have exactly one zero such that $cx_0 < 0$. Then $x_0 = \frac{-b}{a}e^{cx_0}$ or $cx_0 = \frac{-bc}{a}e^{cx_0}$. But $cx_0 < 0$ implies that $abc > 0$.

Conversely, let $abc > 0$. Then $\frac{c}{a}\left[y(x)\right] = cx + \frac{bc}{a}e^{cx}$, which is zero if and only if $y(x) = 0$. If $v = cx$, then $g(v) = v + \frac{bc}{a}e^{v}$. Since it is given that $\frac{bc}{a} > 0$, any value of v that would satisfy $g(v) = 0$ must be less than zero. That is, $v_0 = cx_0 < 0$ if such a v exists. If v_0 is a zero for g, then, from theorem 1, $|v_0| < \left|\frac{bc}{a}\right|$. From theorem 2, there is at most one zero. Observe that

$$\left[g(0)\right]\left[g\left(\frac{-bc}{a}\right)\right] = \frac{bc}{a}\left(\frac{-bc}{a} + \frac{bc}{a}e^{\frac{-bc}{a}}\right)$$

$$= -\left(\frac{bc}{a}\right)^2 + \left(\frac{bc}{a}\right)^2 e^{\frac{-bc}{a}}$$

$$< 0$$

Thus, by the intermediate value theorem, there exists a zero for $g(v)$ and hence for $y(x)$.

EXAMPLES

The theorems above address all transcendental functions which can be expressed in the form $y = ax + be^{cx}$. The following examples should clarify the application of these theorems.

Example 1: Find the intervals of the zeros of $y = -6x + 3e^{2x}$.

Solution: From theorem 5, this function has no zero.

Example 2: Find the intervals of the zeros of $y = -4x + e^{x}$.

Solution: From theorem 3, y has two zeros. If y has two zeros, then from theorem 2 they must be positive. Also from theorem 2, the first zero lies in the interval $\left(\frac{1}{4}, 1\right)$ and the second lies in the interval (1, 4.16).

Example 3: Find the interval containing the zeros for $y = 4x - 2e^{-2x}$.

Solution: y has a single zero (from theorem 6), and its sign is positive. From theorem 1, the zero is in the interval $\left(\frac{1}{2}, \frac{\ln 2}{6}\right)$, which equals $\left(\frac{1}{2}, 0.12\right)$.

ALTERNATE FORMS

Example 4: Show that the zeros of $y = ae^{rt} + bte^{st}$ (where r, t, and s are arbitrary real numbers) are the zeros of a function $g = bt + ae^{dt}$.

Solution: If y has a zero t_0, we can write $ae^{rt_0} = -bt_0e^{st_0}$. Dividing both sides by e^{st_0}, $ae^{(r-s)t_0} = -bt_0$ or $ae^{dt_0} + bt_0 = 0$. That is, t_0 is a zero of $g = ae^{dt} + bt$ and hence of y.

Example 5: Show that by a suitable transformation, the function $y = ax + be^{cx} + d$ can be reduced to a function $g = tu + re^{u}$ and that thus the zeros of g lead directly to those of y.

Solution: Set $u = cx + \frac{cd}{a}$ so that

$$ax + be^{cx} + d = a\left(\frac{u}{c} - \frac{d}{a}\right) + be^{c\left(\frac{u}{c} - \frac{d}{a}\right)} + d$$

which equals $g(u)$, or

$$g(u) = \frac{au}{c} - d + be^{u-\frac{cd}{a}} + d$$

or

$$g(u) = \frac{au}{c} + be^{u-\frac{cd}{a}}$$

$$= tu + re^{u}$$

where $r = be^{-\frac{cd}{a}}$ and $t = \frac{a}{c}$.

Let u_0 be a zero for g; then, by using the above theorems on g, we can locate u_0. Thus, if u_0 is a zero, then

$$a\left(\frac{u_0}{c} - \frac{d}{a}\right) + be^{c\left(\frac{u_0}{c} - \frac{d}{a}\right)} + d = 0$$

and $\frac{u_0}{c} - \frac{d}{a}$ is a zero for y. But $x_0 = \frac{u_0}{c} - \frac{d}{a}$. Thus, by the inverse transformation, one can find x_0 directly from u_0.

Example 6: Show that the function $y = ax + b \ln (cx + d) + p$ (where p is an arbitrary real number) can be reduced to a function $g(x) = cx + ke^{sx} + d$ $\left(\text{where } k = -e^{-\frac{p}{b}} \text{ and } s = -\frac{a}{b}\right)$ whose zeros are those of y.

Solution: Let x_0 be a zero for y. Then

$$ax_0 + b \ln (cx_0 + d) + p = 0$$

or

$$\frac{-ax_0 - p}{b} = \ln (cx_0 + d)$$

or

$$e^{\frac{-ax_0 - p}{b}} = cx_0 + d$$

and

$$\left(e^{-\frac{p}{b}}\right)e^{-\frac{a}{b}x_0} = cx_0 + d$$

Hence,

$$cx_0 - e^{-\frac{p}{b}}e^{-\frac{a}{b}x_0} + d = 0$$

or

$$cx_0 + ke^{sx_0} + d = 0$$

Thus, a zero for g is one for y.

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16. Abstract

The real zeros of the transcendental function $y = ax + be^{cx}$ are characterized, and the results should alleviate the difficulty of determining their existence, location, and number.

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