A METHOD OF CHARACTERISTICS
FOR STEADY THREE-DIMENSIONAL
SUPersonic FLOW WITH APPLICATION
TO INCLINED BODIES OF REVOLUTION

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Characteristics theory for three-dimensional compressible flow is reviewed and the compatibility equations are derived in terms of pressure and stream angles as dependent variables. The relative merits of both characteristic and reference plane methods are discussed. A reference plane method is described which employs a uniformly spaced finite difference mesh between the body and shock surfaces. Numerical procedures for differentiation and interpolation ensure second-order accuracy in terms of mesh spacing.

Results for the flow around a blunt, circular cone, both blunted and pointed, are presented to demonstrate the method described. Predictions of surface and jet pressures are in good agreement with available experimental results for a 15° sphere-cone at 10° angle of attack. Bluntness effects on shock-layer properties far from the nose are reasonably well predicted.
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SUMMARY

Characteristics theory for three-dimensional compressible flow is reviewed and the compatibility equations are derived in terms of pressure and stream angles as dependent variables. The relative merits of bicharacteristic and reference plane methods are discussed. A reference plane method is developed and demonstrated for pointed and blunted bodies at angle of attack.

The fundamental complications arising in a three-dimensional method of characteristics are that: (1) the compatibility equations contain "cross-derivatives" in a noncharacteristic direction; and (2) there is an increased need for interpolation to prevent the computed data surfaces from becoming skewed. These problems are minimized by using a reference plane method with a prescribed and uniform spacing of mesh points. The finite difference mesh employed consists of an equal number of points between the body and shock surfaces in each reference plane. Numerical procedures for differentiation and interpolation ensure second-order accuracy in terms of mesh spacing. Fourier analysis is employed in the circumferential direction and is found to be effective in reducing the number of reference planes and computing times. A typical mesh consists of 7 planes with 15 points in each plane. The unit computation time is about 0.0013 minute per point on an IBM 7094 computer.

Results for the flow around inclined, circular cones, both blunted and pointed, are presented to demonstrate the method described. Predictions of surface and pitot pressures are in good agreement with available experimental results for a 15° sphere-cone at 10° angle of attack. Bluntness effects on shock-layer properties far from the nose are reasonably well predicted.

INTRODUCTION

The method of characteristics has been well known for many years and excellent theoretical developments can be found in numerous texts and reports of which references 1 through 3 give the most complete reviews of multidimensional theory. However, until recent years there have been relatively few practical applications of the methods to three-dimensional flows. This is probably due partly to the large number of operations involved in finite difference calculations in three dimensions, and partly to the extra degree of freedom that results from the existence of characteristic surfaces rather than
characteristic lines. High-speed computers have made it technically feasible to perform calculations for three-dimensional steady and unsteady flows and several groups have reported such work. For example, many recent variations on the method of characteristics are reported in references 4 through 10, and a successful use of a noncharacteristic finite difference method is described in reference 11. A number of the proposed characteristics schemes have been described and discussed critically in reference 12.

All the proposed difference schemes for three-dimensional characteristics can be placed in one of two broad categories: (1) reference plane methods (called semi-characteristic methods in ref. 12); and (2) bicharacteristic methods. In reference plane methods the characteristic lines are obtained from the projection of Mach cones and streamlines onto a prescribed plane. In bicharacteristic methods the characteristic lines are the generators of the Mach cone and the actual streamlines. Since, in three dimensions, the equations written along these characteristic lines are partial differential equations, a numerical evaluation of "cross-derivatives" off the characteristic lines is necessary. (In this sense, three-dimensional characteristic methods are similar to noncharacteristic methods for two-dimensional flow.) The problem encountered with most bicharacteristic methods arises from the need to employ a two-dimensional array of data to evaluate the cross-derivatives and to perform the required interpolation in the initial data surface. Furthermore, if a conventional characteristics mesh is used, the distribution of mesh points in the data surfaces tends to become very nonuniform, which causes additional complications. For these reasons one is led to reference plane methods, where a better control can be maintained on the finite difference mesh and where curve fits can be made with respect to a single variable. Reference plane methods have been criticized on the theoretical grounds that the initial data may be outside the domain of dependence of the calculated point. However, problems related to this criticism have not materialized. In fact, the use of such data is precisely what is required by the well-known Courant-Friedrichs-Lewey stability condition.

In the present work, the compatibility equations of characteristic theory are derived for both the bicharacteristic and reference plane methods. Practical difficulties with the bicharacteristic method are discussed and a finite difference scheme based on the reference plane method is proposed. The proposed method abandons the usual characteristic mesh and employs uniformly spaced points along lines lying in equally spaced meridional planes. Numerical interpolation and differentiation are accomplished by means of second-degree polynomials in the radial direction and by Fourier analysis in the circumferential direction.

Preliminary results by the present method were described in reference 13 and compared with calculations by the method of reference 4. Extensive comparisons with experiment were shown in reference 14, establishing the reliability of the numerical methods. In the present report the flow equations and numerical techniques are described in greater detail than was possible in references 13 and 14.
PRINCIPAL SYMBOLS

\( a \)  
speed of sound

\( C_1, C_2, C_3 \)  
bicharacteristic directions

\( C_1^*, C_2^* \)  
characteristic directions in meridional planes

\( C_p \)  
pressure coefficient

\( C_{pp} \)  
pitot-pressure coefficient

\( \hat{e}_x, \hat{e}_r, \hat{e}_\phi, \hat{X}_i \)  
unit vectors along \( x, r, \phi \)

\( h \)  
enthalpy

\( H \)  
total enthalpy

\( k \)  
smoothing constant (eq. (84))

\( \bar{K} \)  
coefficient of numerical diffusion term (eq. (84))

\( M \)  
Mach number

\( \hat{N}, \hat{L}, \hat{T} \)  
shock normal and tangent vectors

\( p \)  
pressure

\( R_n \)  
body nose radius

\( s^* \)  
projection of streamlines on meridional planes

\( s, n, t \)  
streamline coordinates

\( u, v, w \)  
velocity components along \( \hat{e}_x, \hat{e}_r, \hat{e}_\phi \)

\( V \)  
total velocity \( \vec{V} = V \hat{s} = V \hat{x}_1 \)

\( x \)  
axial distance from blunt nose (body axis)

\( \hat{x}_i \)  
unit vectors along \( s, n, t \)

\( x, r, \phi \)  
cylindrical coordinates

\( \hat{y}_i \)  
unit vectors along characteristic coordinates

\( \hat{z}_i \)  
unit vectors along \( \xi, n, \zeta \)

\( \alpha \)  
angle of attack, deg
\[ \alpha_{ij} \] transformation matrix (eq. (16))
\[ \beta = \sqrt{M^2 - 1} \]
\[ \gamma \] specific heat ratio
\[ \delta \] shock angle in cross plane (eq. (A6))
\[ \xi_{ij} \] transformation matrix (eq. (A18))
\[ \xi, \eta, \zeta \] nonorthogonal shock-layer coordinates (fig. 3)
\[ \theta \] flow angle from x axis in meridional plane, \( \tan^{-1}(v/u) \) (see fig. 1)
\[ \lambda \] angle between \( \hat{e}_r \) and \( \hat{n} \)
\[ \mu \] Mach angle, \( \sin^{-1}(1/M) \)
\[ \mu^* \] projection of \( \mu \) on meridional plane
\[ \nu_{ij} \] transformation matrix (eq. (A13))
\[ \rho \] density
\[ \sigma \] shock angle in meridional plane (eq. (A5))
\[ \phi \] crossflow angle, \( \sin^{-1} w/V \) (see fig. 1)
\[ \Phi \] azimuthal angle, cylindrical coordinates (see fig. 1), deg

Subscripts

\( A, B \) initial and new data lines
\( b \) body
\( j, r \) indices for radial position of mesh points
\( (j = 1, 2, \ldots, J - 1, J) \) (see fig. 5)
\( k \) index for circumferential position of mesh point
\( (k = 1, 2, \ldots, L - 1, L) \)
\( n \) body nose
\( s \) shock
\( \infty \) free-stream condition
Superscripts

\( i \) index for axial position of mesh point

\( \nu \) index denoting variables \( p, \theta, \phi, \rho \) for \( \nu = 1, 2, 3, 4 \)

\( * \) quantity or component referenced to meridional plane

\( \w \) unit vector

\( \rightarrow \) general vector

Note: See equation (14) for definition of index notation.

THEORY

Basic Equations of Inviscid Flow

The derivation of the characteristic relations will be carried out in this section starting with the equations for inviscid equilibrium flow written in their intrinsic form with pressure and flow angles as dependent variables. This choice of variables eliminates entropy derivatives from the flow equations, thereby simplifying the analysis. Following the development of reference 15, the combined continuity and momentum equations are written in vector form.

\[
\frac{\beta^2}{\rho V^2} \hat{s} \cdot \text{grad} \, p + \text{div} \, \hat{s} = 0 \quad (1)
\]

\[
\frac{1}{\rho V^2} \hat{n} \cdot \text{grad} \, p + \hat{t} \cdot \text{curl} \, \hat{s} = 0 \quad (2)
\]

\[
\frac{1}{\rho V^2} \hat{t} \cdot \text{grad} \, p - \hat{n} \cdot \text{curl} \, \hat{s} = 0 \quad (3)
\]

where \( \hat{s}, \hat{n}, \hat{t} \) are orthogonal unit vectors with \( \hat{s} \) tangent to the streamlines. For rotational, inviscid, nonheat-conducting, and nonreacting flow, entropy is conserved along streamlines and this requires

\[
\hat{s} \cdot \text{grad} \, p - a^2 \hat{s} \cdot \text{grad} \, \rho = 0 \quad (4)
\]
Conservation of total enthalpy everywhere gives

\[ h + \frac{V^2}{2} = h_t = \text{constant} \]  \hspace{1cm} (5)

Finally, the equation of state may be written as

\[ h = h(p, \rho) \] \hspace{1cm} (6a)

or

\[ a = a(p, \rho) \] \hspace{1cm} (6b)

In the equations above, if \( \hat{n} \) is chosen to lie in meridional planes through the body axis, the unit vectors can be written as follows in terms of flow deflection angles \( \theta \) and \( \phi \) (see fig. 1).
\[ \hat{s} = \cos \phi \cos \theta \hat{e}_x + \cos \phi \sin \theta \hat{e}_r + \sin \phi \hat{e}_\phi \] (7)

\[ \hat{n} = -\sin \theta \hat{e}_x + \cos \theta \hat{e}_r \] (8)

\[ \hat{t} = -\sin \phi \cos \theta \hat{e}_x - \sin \phi \sin \theta \hat{e}_r + \cos \phi \hat{e}_\phi \] (9)

Using equations (7) through (9) with the standard vector formulas in cylindrical coordinates, one can obtain the following from equations (1) through (4):

\[ \frac{\beta^2}{\rho y^2} \frac{\partial p}{\partial s} + \cos \phi \frac{\partial \theta}{\partial n} + \frac{\partial \phi}{\partial t} = -\frac{\cos \phi \sin \theta}{r} \] (10)

\[ \frac{1}{\rho y^2} \frac{\partial p}{\partial n} + \cos \phi \frac{\partial \theta}{\partial s} \frac{\sin^2 \phi \cos \theta}{r} \] (11)

\[ \frac{1}{\rho y^2} \frac{\partial p}{\partial t} + \frac{\partial \phi}{\partial s} = -\frac{\sin \phi \sin \theta}{r} \] (12)

\[ \frac{\partial \rho}{\partial s} = \frac{1}{a^2} \frac{\partial p}{\partial s} \] (13)

**Characteristics Theory**

Notation.- Characteristics theory is most conveniently developed, especially for the multidimensional case, if one uses index notation. Therefore, the notation and development of Courant and Friedrichs (ref. 1, p. 75) will be followed in this review. Summation convention is used wherein a summation symbol is implied by a repeated index. Thus equations (10) to (12) may be written simply as

\[ a_{\mu \nu} \frac{\partial u^\nu}{\partial x_1} = f_\mu \] (14)
where the indices refer to:

\( v \) dependent variable \((p, \theta, \phi\) for \( v = 1, 2, 3\))

\( i \) independent variable \((s, n, t\) for \( i = 1, 2, 3\))

\( \mu \) equation (eqs. (10)-(12) for \( \mu = 1, 2, 3\))

Coordinate transformation.- Equation (14) can be expressed in terms of new coordinate directions \( y_j \) obtained by a rotation of the original coordinates. Thus one can write

\[
\hat{x}_i = a_{ij} \hat{y}_j
\]  

(15)

where \( \hat{x}_i \) and \( \hat{y}_j \) are unit vectors along \( x_i \) and \( y_j \), respectively, and the elements of the transformation matrix are the direction cosines defined as follows in terms of the scalar products of \( \hat{x}_i \) and \( \hat{y}_j \):

\[
a_{ij} = \begin{pmatrix}
\hat{x}_1 \cdot \hat{y}_1 & \hat{x}_1 \cdot \hat{y}_2 & \hat{x}_1 \cdot \hat{y}_3 \\
\hat{x}_2 \cdot \hat{y}_1 & \hat{x}_2 \cdot \hat{y}_2 & \hat{x}_2 \cdot \hat{y}_3 \\
\hat{x}_3 \cdot \hat{y}_1 & \hat{x}_3 \cdot \hat{y}_2 & \hat{x}_3 \cdot \hat{y}_3
\end{pmatrix}
\]  

(16)

With this transformation, equation (14) becomes

\[
b^j_{\mu \nu} \frac{\partial u^\nu}{\partial y_j} = f^\mu
\]  

(17)

where

\[
b^j_{\mu \nu} = a^i_{\mu \nu} a_{ij}
\]  

(18)

Characteristic directions and compatibility equations.- If initial data are given for \( u^\nu \) on the surface \( y_1 = 0 \), equation (17) can be solved for \( \frac{\partial u^\nu}{\partial y_1} \) in order to generate data on an adjoining surface \( y_1 = \Delta y_1 \). Placing derivatives with respect to \( y_2 \) and \( y_3 \) on the right side of equation (17), one obtains

\[
b^{(1)}_{\mu \nu} \frac{\partial u^\nu}{\partial y_1} = g_\mu
\]  

(19)
where

\[ g_{\mu} = f_{\mu} - b^{k}_{\mu \nu} \frac{\partial u^{\nu}}{\partial y_{k}} \quad (k = 2, 3) \]

Equation (19) can be considered as a set of algebraic equations for \( \frac{\partial u^{\nu}}{\partial y_{1}} \), with a solution by Cramer's rule giving

\[ \frac{\partial u^{\nu}}{\partial y_{1}} = \frac{D^{\nu}}{b_{\nu}^{(1)}} \]

(20)

where \( \left| b_{\mu \nu}^{(1)} \right| \) is the determinant of \( b_{\mu \nu}^{(1)} \) and \( D^{\nu} \) is the determinant obtained by replacing the \( \nu \)th column of \( b_{\mu \nu}^{(1)} \) with \( g_{\mu} \).

If \( \left| b_{\mu \nu}^{(1)} \right| \) vanishes, then \( y_{1} \) is normal to a characteristic surface and the flow equations give no information about derivatives in this direction; that is, \( \frac{\partial u^{\nu}}{\partial y_{1}} \) may be discontinuous. Therefore, the equation of the characteristic surface is given by

\[ \left| b_{\mu \nu}^{(1)} \right| = 0 \]

(21)

On the other hand, if equation (21) is satisfied, then the numerator of equation (20) must also vanish in order for a solution to exist. Therefore,

\[ D^{\nu} = 0 \quad (\nu = 1, 2, 3) \]

(22)

gives the so-called compatibility equations. It may be noted from equation (19) that the compatibility relations contain one less dimension than the original differential equations. For the three-dimensional problem the compatibility equations involve derivatives in two directions.

Bicharacteristic Method

The characteristic relations reviewed in the previous section can now be specialized to the problem of equilibrium three-dimensional gas flow. Consider equations (10) through (13) with the supplementary conditions (5) and (6). Note that equation (13) is already in characteristic form since it involves derivatives in one direction; the streamline, \( s \), is a line across which the density gradient, \( \partial \rho / \partial n \), may be discontinuous. Therefore, one need only consider equations (10), (11), and (12) for which the \( i \) coefficient matrices may be written
where $\delta_{ki}$ is the Kronecker delta

$$
\delta_{ki} = \begin{cases} 
0 & k \neq i \\
1 & k = i 
\end{cases}
$$

The transformed coefficient matrix given by (18) becomes

$$
b^{(j)}_{\mu\nu} = \begin{pmatrix}
a_{1j} & \frac{\beta^2}{\rho V^2} & a_{2j} \cos \phi & a_{3j} \\
a_{2j} & \frac{1}{\rho V^2} & a_{1j} \cos \phi & 0 \\
a_{3j} & \frac{1}{\rho V^2} & 0 & a_{ij}
\end{pmatrix}
$$

Characteristic cone.- The characteristic determinant, equation (21), can now be evaluated to give the equation of the characteristic surface in terms of the coefficients of the governing differential equations. Using equation (24) one obtains

$$
|b^{(1)}_{\mu\nu}| = \left( \frac{a_{21}}{a_{11}} \right)^2 + \left( \frac{a_{31}}{a_{11}} \right)^2 - \beta^2 = 0
$$

where $\beta^2 = M^2 - 1$. Now, from equation (16) the terms in equation (25) are recognized as

$$
a_{11} = \hat{x}_1 \cdot \hat{y}_1 \quad a_{21} = \hat{x}_2 \cdot \hat{y}_1 \quad a_{31} = \hat{x}_3 \cdot \hat{y}_1
$$
which are the components of \( \hat{y}_1 \) along the \( \hat{x}_1 \) coordinates. Thus, equation (25) describes a cone around the \( x_1 \) or \( s \) axis, as indicated in figure 2(a), making the angle \( 90 - \nu \) with the \( s \) direction. This cone is normal to the characteristic cone. The vector \( \hat{y}_1 \) lies along a generator of the normal cone, and the vanishing determinant (21) means that the derivatives with respect to \( y_1 \) may be discontinuous; that is, the differential equations (1), (2), and (3) cannot give any information about derivatives in this direction.

It follows, then, if the \( y_i \) coordinates are orthogonal, that \( \hat{y}_2 \) and \( \hat{y}_3 \) are tangent to the characteristic cone. If \( \hat{y}_2 \) is chosen to lie along a generatrix of the cone, then \( y_2 \) is called a bicharacteristic direction.

**Compatibility equations.** Any one of the three conditions given by equation (22) can be used to obtain the compatibility equations. Consider the case for \( \nu = 1 \).

\[
D_1 = \begin{vmatrix} g_1 & \alpha_{21} \cos \phi & \alpha_{31} \\ g_2 & \alpha_{11} \cos \phi & 0 \\ g_3 & 0 & \alpha_{11} \end{vmatrix} = 0
\]  

(26)

Expanding (26) one obtains

\[
g_1 - (\alpha_{21}/\alpha_{11})g_2 - (\alpha_{31}/\alpha_{11})g_3 = 0
\]

(27)

where

\[
g_1 = f_1 - \frac{\beta^2}{\rho V^2} \left( \alpha_{12} \frac{\partial p}{\partial y_2} + \alpha_{13} \frac{\partial p}{\partial y_3} \right) - \cos \phi \left( \alpha_{22} \frac{\partial \theta}{\partial y_2} + \alpha_{23} \frac{\partial \theta}{\partial y_3} \right) - \left( \alpha_{32} \frac{\partial \phi}{\partial y_2} + \alpha_{33} \frac{\partial \phi}{\partial y_3} \right)
\]

(28)

\[
g_2 = f_2 - \frac{1}{\rho V^2} \left( \alpha_{22} \frac{\partial p}{\partial y_2} + \alpha_{23} \frac{\partial p}{\partial y_3} \right) - \cos \phi \left( \alpha_{12} \frac{\partial \theta}{\partial y_2} + \alpha_{13} \frac{\partial \theta}{\partial y_3} \right)
\]

(29)
The bicharacteristic direction $y_2$ can be arbitrarily chosen to lie along any ray of the characteristic cone. This means that an infinite number of equations can be obtained from (27). However, three equations are sufficient to determine the solution for the three dependent variables $p$, $\theta$, $\phi$.

The bicharacteristics can be chosen so as to simplify the compatibility equations. First it is noted from equation (16), identifying $(x_1, x_2, x_3)$ with $(s, n, t)$, that

$$\alpha_{ij} = \begin{pmatrix} s \cdot \hat{y}_1 & s \cdot \hat{y}_2 & s \cdot \hat{y}_3 \\ \hat{n} \cdot \hat{y}_1 & \hat{n} \cdot \hat{y}_2 & \hat{n} \cdot \hat{y}_3 \\ \hat{t} \cdot \hat{y}_1 & \hat{t} \cdot \hat{y}_2 & \hat{t} \cdot \hat{y}_3 \end{pmatrix}$$

Thus a number of the elements of $\alpha_{ij}$ will be zero if $\hat{y}_2$ is in the $s$-$n$ plane, and $\hat{y}_3$ lies along the $\hat{t}$ axis. In this case (see fig. 2(b)) there are two possibilities given by

$$\alpha_{ij} = \begin{pmatrix} \sin \mu & \cos \mu & 0 \\ \mp \cos \mu & \pm \sin \mu & 0 \\ 0 & 0 & \mp 1 \end{pmatrix}$$

where the upper sign refers to the left-running characteristic $C_1$, and the lower sign to the right-running characteristic $C_2$.

Letting $\hat{y}_2$ lie in the $s$-$t$ plane so that $\hat{y}_3$ lies along the $\hat{n}$ axis gives

$$\alpha_{ij} = \begin{pmatrix} \sin \mu & \cos \mu & 0 \\ 0 & 0 & -1 \\ -\cos \mu & \sin \mu & 0 \end{pmatrix}$$

where the signs correspond to the bicharacteristic direction $C_3$ which increases with increasing $s$ and $t$ as shown in figure 2(c).

---

1Redundant schemes which make use of more than three equations are discussed in references 8, 10, 12, and 16.

Substitution of $a_{ij}$ from equations (31) and (32) into equation (27) results in three equations along the bicharacteristics $C_1$, $C_2$, and $C_3$. Using equation (31) for $C_1$ and $C_2$, one obtains

$$g_1 \pm \cot \mu g_2 = 0$$

where $y_2 = C_1$ for the upper sign and $y_2 = C_2$ for the lower sign. Similarly, one obtains from equation (32) the following equation for the direction $C_3$:

$$g_1 + \cot \mu g_3 = 0$$

Straightforward substitution of the $g_i$ defined in equations (28) to (30) with appropriate elements of $a_{ij}$ from equation (31) or equation (32) yields, with some rearrangement, the following compatibility equations:

$$\frac{\beta}{\rho V^2} \frac{\partial P}{\partial C_1} + \cos \phi \frac{\partial \theta}{\partial C_1} = (f_1 + \beta f_2 - \frac{\partial \phi}{\partial t}) \sin \mu$$  \hspace{1cm} (33)

$$\frac{\beta}{\rho V^2} \frac{\partial P}{\partial C_2} - \cos \phi \frac{\partial \theta}{\partial C_2} = (f_1 - \beta f_2 - \frac{\partial \phi}{\partial t}) \sin \mu$$  \hspace{1cm} (34)

$$\frac{\beta}{\rho V^2} \frac{\partial P}{\partial C_3} + \frac{\partial \phi}{\partial n} = (f_1 + \beta f_3 - \cos \phi \frac{\partial \theta}{\partial n}) \sin \mu$$  \hspace{1cm} (35)

where, from the right side of equations (10) through (12),

$$f_1 = - \frac{\cos \phi \sin \theta}{r}, \quad f_2 = \frac{\sin^2 \phi \cos \theta}{r}, \quad \text{and} \quad f_3 = - \frac{\sin \phi \sin \theta}{r}$$

For two-dimensional flow ($\phi = 0$), equations (33) and (34) reduce to the usual compatibility equations and equation (35) becomes the streamwise momentum equation.

Fundamental complications.- In comparison with axially symmetric flows, equations (33) through (35) have two complicating features which were mentioned earlier. These are (1) the presence of "cross-derivatives" $\partial \phi/\partial t$ and $\partial \theta/\partial n$ on the right side, and (2) the need to perform a two-parameter interpolation for data in the initial data surface. The second problem arises
because bicharacteristics through a general mesh point (in a prescribed surface) do not, in general, pass through mesh points in the initial data surface.²

Many schemes have been proposed using equations of this form along bicharacteristics (see, e.g., refs. 5-8 and 12). However, the programming of such methods tends to be cumbersome, and the accuracy of evaluating the cross-derivatives is in most cases less than that of the basic calculations. (An exception is the recent work reported in reference 10.) These problems are minimized if characteristics lying in prescribed reference planes are employed. Then the interpolation for initial data and evaluation of cross-derivatives can be reduced to a set of one-parameter curve fits with second-order accuracy. In the next section, the compatibility equations will be derived for characteristic lines which are the projections of bicharacteristics \( C_1, C_2, C_3 \) onto reference planes.

Reference Plane Method

In the following development, the flow equations are written in terms of two components lying in predetermined reference planes and a third component directed out of these planes. For most problems encountered in external aerodynamics it is convenient to specify the reference planes as the axial planes, \( \phi = \text{constant} \), of a cylindrical coordinate system (see fig. 1). The solution of problems with axial symmetry is determined by calculation along a single plane, but three-dimensional problems require calculation along several planes simultaneously. Characteristic theory is employed to calculate the flow along each plane. To achieve this, the compatibility equations along the projections of the bicharacteristics on the reference planes must be derived. The procedure will be to find the pertinent characteristics and compatibility relations from the intrinsic momentum equations applicable to the reference planes. First, however, it will be necessary to discuss the coordinate mesh which will be used to describe the shock layer.

Finite difference mesh.- The cylindrical coordinate system used to expand the vector relations in equations (1) through (3), and to define the reference planes, is not ideal from the computational standpoint. This stems from the fact that special treatment would be required for bodies with non-circular cross sections and, more importantly, for shock surfaces. One is therefore led to a finite difference mesh which divides the shock layer into a number of annular rings which include both the shock and body surfaces, as shown in figure 3. The resulting mesh points are connected by a nonorthogonal \( \xi, \eta, \zeta \) system of coordinates: \( \xi \) and \( \eta \) lie in the reference plane with \( \eta \) usually normal to the body surface; \( \zeta \) is directed out of but not generally normal to the reference plane.

²Conversely, bicharacteristics through known points on a plane initial data surface will, in general, intersect at points lying in a nonplanar surface. The subsequent data surfaces become increasingly distorted as a computation proceeds away from the initial data plane.
A related system is one in which \( \xi \) is replaced locally by the projection, \( s^* \), of streamlines onto the reference planes. It is this \( s^* \), \( \eta \), \( \xi \) system which is used to express the intrinsic flow equations (10) through (13) in a form needed for the present reference plane analysis. The unit vectors, \( \hat{x}_i = (\hat{s}, \hat{n}, \hat{t}) \), in streamline coordinates are related to the new system by direction cosines, \( \varepsilon^*_{ij} \), defined by

\[
\hat{x}_i = \varepsilon^*_{ij} \hat{z}_j
\]  

where \( \hat{z}_j = (\hat{s}^*, \hat{n}^*, \hat{\xi}^*) \).

Written in terms of its components, equation (36) gives

\[
\hat{s} = \varepsilon^*_{11} \hat{s}^* + \varepsilon^*_{12} \hat{n}^* + \varepsilon^*_{12} \hat{\xi}^*
\]  

\[
\hat{n} = \varepsilon^*_{21} \hat{s}^* + \varepsilon^*_{22} \hat{n}^* + \varepsilon^*_{23} \hat{\xi}^*
\]  

\[
\hat{t} = \varepsilon^*_{31} \hat{s}^* + \varepsilon^*_{32} \hat{n}^* + \varepsilon^*_{33} \hat{\xi}^*
\]

Appendix A describes how \( \varepsilon^*_{ij} \) is calculated in terms of the body and shock-wave shapes. The detailed expressions for the direction cosines are not needed for the development of this section but it should be noted that, since the shock shape itself is obtained from a solution of the problem (i.e., a direct as opposed to an inverse approach), \( \varepsilon^*_{ij} \) is not known beforehand for the entire flow. However, in a locally supersonic region, where the shock can be calculated step by step, \( \varepsilon^*_{ij} \) can always be determined as the calculation proceeds downstream from an initial data line.

Reference plane equations.- Recalling the rules for a directional derivative, equations (37) and (39) yield the following expressions for derivatives in the \( s \) and \( t \) directions.
These differentiation rules allow one to write the intrinsic flow equations in terms of the desired planar components. Substituting equations (40) and (41) into equations (10) through (13) and regrouping terms, one obtains

\[ \frac{\partial}{\partial s} = \varepsilon_{11} \frac{\partial}{\partial s} + \varepsilon_{12} \frac{\partial}{\partial \eta} + \varepsilon_{13} \frac{\partial}{\partial \zeta} \]  
(40)

\[ \frac{\partial}{\partial t} = \varepsilon_{31} \frac{\partial}{\partial s} + \varepsilon_{32} \frac{\partial}{\partial \eta} + \varepsilon_{33} \frac{\partial}{\partial \zeta} \]  
(41)

The left sides of these equations contain derivatives in the reference planes and the remaining terms are all lumped into the \( f_1^* \), which are given by

\[ f_1^* = - \cos \phi \sin \theta \left[ \frac{1}{\rho V^2} \left( \varepsilon_{12} \frac{\partial p}{\partial \eta} + \varepsilon_{13} \frac{\partial p}{\partial \zeta} \right) + \left( \varepsilon_{31} \frac{\partial \phi}{\partial s} + \varepsilon_{32} \frac{\partial \phi}{\partial \eta} + \varepsilon_{33} \frac{\partial \phi}{\partial \zeta} \right) \right] \]  
(46)

\[ f_2^* = \frac{\sin^2 \phi \cos \theta}{r} - \cos \phi \left( \varepsilon_{12} \frac{\partial \phi}{\partial \eta} + \varepsilon_{13} \frac{\partial \phi}{\partial \zeta} \right) \]  
(47)

\[ f_3^* = \frac{1}{\varepsilon_{11}} \left[ \frac{\sin \phi \sin \theta}{r} - \frac{1}{\rho V^2} \left( \varepsilon_{31} \frac{\partial p}{\partial s} + \varepsilon_{32} \frac{\partial p}{\partial \eta} + \varepsilon_{33} \frac{\partial p}{\partial \zeta} \right) \right] - \left( \varepsilon_{12} \frac{\partial \phi}{\partial \eta} + \varepsilon_{13} \frac{\partial \phi}{\partial \zeta} \right) \]  
(48)

\[ f_4^* = \frac{1}{a^2} \frac{\partial p}{\partial s} + \frac{1}{\varepsilon_{11}} \left[ \varepsilon_{12} \left( \frac{1}{a^2} \frac{\partial p}{\partial \eta} - \frac{\partial p}{\partial \eta} \right) + \varepsilon_{13} \left( \frac{1}{a^2} \frac{\partial p}{\partial \zeta} - \frac{\partial p}{\partial \zeta} \right) \right] \]  
(49)
The $f_1^*$ thus expressed contain derivatives along the $\eta$ and $\zeta$ coordinate directions and, as a result, can be easily evaluated with standard one-parameter differentiation routines. It will also be seen, when the finite difference scheme is described, that the derivatives $\partial\phi/\partial s^*$ appearing in $f_1^*$ and $\partial p/\partial s^*$ appearing in $f_3^*$ and $f_4^*$ are easily treated.

It is important to note that, although written in a simplified form, equations (42) through (45) contain no additional approximations beyond those in the original equations. Only the usual assumptions pertaining to viscosity and heat conduction are made. The bracketed [ ] terms in the $f_i^*$ all vanish for axially symmetric flows (i.e., for $\phi = 0$), and the equations reduce to the familiar intrinsic equations for zero angle of attack.

Characteristic directions.- Characteristics theory, as outlined in a previous section, can now be applied to equations (42) through (45). It is first observed that equations (44) and (45) are already in the desired characteristic form. These equations give no information about the normal derivative $\partial/\partial n$; therefore, $s^*$ is a characteristic direction for $\phi$ and $\rho$. This is in contrast to equations (42) and (43), which can be combined to give derivatives in different directions. Writing the latter in the form of equation (14), one has

$$a^i_{\mu\nu} \frac{\partial u^\nu}{\partial x^*_i} = f^*_\mu$$

Expressed in terms of new coordinates, $y^*_i$, equation (50) becomes (cf. eqs. (17) and (24)):

$$b^*_{\mu\nu} \frac{\partial u^\nu}{\partial y^*_j} = f^*_\mu$$

where

$$b^*_{\mu\nu} = \begin{pmatrix} \alpha_{1j} \xi^*_{11} & \frac{\gamma^2}{\rho V^2} & \alpha_{2j} \cos \phi \\ \frac{1}{\rho V^2} & \alpha_{1j} \xi^*_{11} \cos \phi & \alpha_{2j} \cos \phi \\ \end{pmatrix}$$

and where $\alpha_{ij}$ are the elements of the transformation matrix relating $x_i$ and $y_j$. The characteristic directions for equation (51) are obtained from the determinant.
Writing $\alpha_{ij}$ in terms of a rotation angle as in equation (31), one obtains for this case

$$\alpha_{ij} = \begin{pmatrix} \sin \mu^* & \cos \mu^* \\ \mp \cos \mu^* & \pm \sin \mu^* \end{pmatrix}$$

where $\mu^*$ is the angle between the streamline and the characteristics in the reference plane as shown in figure 4. Equation (52) shows that $u^*$ is related to the Mach angle according to

$$\cot \mu^* = \varepsilon_{11}^* \cot \mu$$

**Compatibility equations.** - The compatibility equations applicable to the directions $C_1^*$ and $C_2^*$ can be obtained in the way previously described for the bicharacteristics. For the present case, one has in place of equation (27)

$$g_1^* - \frac{\alpha_{21}}{\alpha_{11}} g_2^* = 0$$

where

$$g_1^* = f_1^* - b_{11}^* \frac{\partial u^{(1)}}{\partial y_2} - b_{12}^* \frac{\partial u^{(2)}}{\partial y_2}$$

and

$$g_2^* = f_2^* - b_{21}^* \frac{\partial u^{(1)}}{\partial y_2} - b_{22}^* \frac{\partial u^{(2)}}{\partial y_2}$$

Algebraic details are straightforward and need not be repeated. The resulting compatibility equations are:
\[
\frac{\beta}{\rho V^2} \frac{\partial p}{\partial C_1^*} + \cos \phi \frac{\partial \theta}{\partial C_1^*} = (f_1^* + \beta f_2^*) \sin \mu^*
\]  \hspace{1cm} (55)

\[
\frac{\beta}{\rho V^2} \frac{\partial p}{\partial C_2^*} - \cos \phi \frac{\partial \theta}{\partial C_2^*} = (f_1^* - \beta f_2^*) \sin \mu^*
\]  \hspace{1cm} (56)

Equations (55) and (56), together with equations (44) and (45), are the final set of differential equations programmed for numerical calculations. They are supplemented by the energy equation (eq. (5))

\[ h + \frac{V^2}{2} = H \]

and the equations of state ((6a) and (6b))

\[ h = h(p, \rho) \]

and

\[ a = a(p, \rho) \]

Details of the numerical methods used in the computer program are described in the next section.

NUMERICAL METHODS

The general theory of characteristics was developed in the previous section, where it was shown that the three-dimensional problem can be reduced to an equivalent two-dimensional form. A numerical solution of the problem can then be accomplished in the usual way by numerically evaluating derivatives in one direction in order to calculate a step forward in the second direction. The problem is analogous to the numerical solution of two-dimensional hyperbolic equations by noncharacteristics methods.

Therefore, in formulating a practical method for calculating three-dimensional flow, the numerical differentiation process is of primary importance. If the differentiation is to be performed efficiently and accurately (i.e., at least second order in a typical mesh dimension), the mesh points should be constrained to lie along simple coordinate lines. Secondly, the boundary calculations are simplified if the coordinates lie on the shock and body surfaces. These considerations suggest the shock-layer-oriented, non-orthogonal coordinates shown in figure 3. The resulting computational procedure is simplified significantly compared with previously proposed three-dimensional characteristics methods (see, e.g., ref. 12).
Presented next are the difference equations, computational logic, and numerical differentiation procedures developed on the basis of this nonorthogonal \((\xi, \eta, \zeta)\) mesh. The problem may be stated as follows:

Given data on an initial \(\eta - \zeta\) surface at \(\xi_0\) where the flow is locally supersonic, it is required to generate, by means of the flow equations and boundary conditions, new data on the adjacent \(\eta - \zeta\) surface at \(\xi_0 + \Delta \xi\).

**Finite Difference Equations**

Figure 5 shows the mesh point arrangement for a typical reference plane \(\phi_k\). Identify with superscripts \((i-1)\) and \((i)\) the initial data line at \(\xi_A\) and new data line at \(\xi_B\). Let the subscripts \(j\) and \(k\) denote the radial and circumferential positions of the mesh points. However, the subscript \(k\) will be omitted for brevity in writing the difference equations.

The method adopted\(^3\) to calculate conditions at a typical mesh point \(i,j\) on \(\xi_B\) makes use of interpolated data at the points of intersection on \(\xi_A\) of the characteristics through point \(i,j\). A three-point Lagrange interpolation is employed to determine the necessary data from known conditions at neighboring mesh points.

Three intersections are required for each field point on \(\xi_B\). To identify these points the convention is adopted whereby the subscript \(\tau\) represents the intersection with \(\xi_A\) of the streamline projection \(s^*\) through point \(i,j\), and the subscripts \(\tau-1\) and \(\tau+1\) represent intersections of characteristics \(C_1^*\) and \(C_2^*\), respectively (see fig. 5).

With this convention the compatibility equations (55) and (56) can be written in the following finite difference form:

\[
A_1 \left( p_j^i - p_{\tau-1}^i \right) + B_1 \left( \theta_j^i - \theta_{\tau-1}^i \right) = F_1 \Delta C_1^* \quad (57)
\]

\[
A_2 \left( p_j^i - p_{\tau+1}^i \right) - B_2 \left( \theta_j^i - \theta_{\tau+1}^i \right) = F_2 \Delta C_2^* \quad (58)
\]

\(^3\)This method is called the Hartree method (ref. 17) and also the inverse method (ref. 12); it was used by Katskova and Chushkin (ref. 9).
Equations (44) and (45) are similarly written as

\[ \phi_j^i - \phi_{j-1}^i = F_3 \Delta s^* \]  
(59)

and

\[ \rho_j^i - \rho_{j-1}^i = F_4 \Delta s^* \]  
(60)

The system of equations is completed by the energy and state equations, (5) and (6). They apply to all of the field points - that is, for the index \( j \) running from \( 2 \) up to \( J-1 \). For the body point, \( j=1 \), equation (57) is replaced by the equation of the body,

\[ r_j^i = f(x_j^i, \phi_k) \]

which permits the calculation of \( \theta_1^i \) by means of equations derived in appendix C. It is shown there that

\[ \tan \theta_1^i = \frac{1}{1 - a^2} \left( \frac{\partial r_1^i}{\partial x} + a \sqrt{b - a^2} \right) \]  
(61a)

where

\[ a = \tan \phi \left( \frac{1}{r_1^i} \frac{\partial r_1^i}{\partial \phi} \right) \quad \text{and} \quad b = 1 + \left( \frac{\partial r_1^i}{\partial x} \right)^2 \]

For bodies of revolution equation (61a) reduces to

\[ \theta_1^i = \tan^{-1} \frac{\partial r_1^i}{\partial x} \]  
(61b)

At the shock, \( j = J \), equations (58), (59), and (60) are replaced by the oblique shock equations. The jump conditions for a general three-dimensional shock surface are developed in appendix B and can be written in the following functional form for fixed free-stream conditions:

\[ (u^v)_j^i = G^v(\sigma^i; \phi_k, \alpha) \]  
(62)
where \( u^v \) represents \( p, \theta, \phi, \rho \) for \( v = 1, 2, 3, 4 \). Here \( \sigma \) and \( \delta \) are the shock-wave angles in the planes \( \phi = \text{constant} \) and \( x = \text{constant} \), respectively, and \( \alpha \) is the angle of attack. Equation (62) depends directly on the unknown shock angle \( \sigma_1 \), and indirectly on the parameters \( \delta_k, \phi_k, \) and \( \alpha \). The shock angle \( \delta_k \) is determined by numerical differentiation of shock coordinates as described in appendix A. The four equations obtained from (62) for \( v = 1 \) through 4, together with equation (57), are sufficient to determine the shock angle \( \sigma_1 \).

### Computational Procedure

**Local iteration.** The difference equations are solved by a standard Euler predictor-corrector method in which the coefficients are treated as constants locally and equal to their average value over the step. An initial guess, say \((u^v)^{i-1}_j = (u^v)^{i-1}_j\), is used to start an iterative procedure by which corrected values of \((u^v)^i_j\) are calculated using coefficients evaluated with the previous predictions. The iteration is continued until the pressure repeats to a specified accuracy. For typical mesh points three correctors reduce the relative error to less than \(1 \times 10^{-5}\). It is shown in standard texts that the iterated result has a truncation error of the order of the step size cubed.

In this method the coefficients \( A_\mu, B_\mu, \) and \( F_\mu \) in equations (57) through (60) are averaged along the characteristic direction appropriate to each equation. For example, the coefficient \( F_1 \) in difference equation (57) represents the average of the right side of differential equation (55), taken along the characteristic \( C_1^*, \) and is written as

\[
F_1 = \frac{1}{2}[(f_1^* + \beta f_2^*)]^{i+1}_j + \frac{1}{2}[(f_1^* + \beta f_2^*)]^{i-1}_\tau-1
\]

in the present notation. Similarly, averages are evaluated using data at point \( \tau + 1 \) for equation (58) and at point \( \tau \) for equations (59) and (60).

**Global iteration.** The set of equations (57) through (60) are solved successively on \( L \) reference planes, \( \phi_k (k = 1, 2, \ldots, L) \). The difference equations governing the flow along various reference planes are coupled by cross-derivatives which are included in the functions \( F_\mu \) appearing on the right side of each equation and by the shock angle \( \delta_k \) appearing in equation (62). In order to solve these equations in an explicit manner, it is therefore necessary first to approximate \( F_\mu \) and \( \delta_k \) with derivatives evaluated on \( \xi_A \). Then, using calculated values on all of the \( L \) planes to evaluate cross-derivatives on \( \xi_B \), one obtains the next approximation to \( F_\mu \) and \( \delta_k \) and the entire process can be repeated. This is referred to as a global iteration, in contrast to the point-by-point iteration employed in the local solution of the difference equations.
Since computing times are generally large in three-dimensional problems (see next section), test cases were run to determine the need for global iteration. Table I shows that excellent results are obtained without iteration, that is, using derivatives evaluated on $\xi_A$. It is therefore expected that this iteration would not be required in most problems.

**Step Size**

Since the mesh points are not constrained to follow characteristic lines with the present method, the size of a forward step, $\Delta \xi = \xi_B - \xi_A$, can be arbitrary to some extent. The only limitation is that, in order to have a stable numerical process, the step should not exceed a certain maximum determined by the region of influence of the initial data. This stability condition will make $\Delta \xi$ depend on the radial step $\Delta \eta$, smaller radial steps requiring smaller forward steps. The lateral step $\Delta \zeta$ does not affect the stability directly, although it does, of course, affect the accuracy. The effect of step size on accuracy is discussed after the following paragraphs on stability condition.

**Stability condition.**—Although the analysis of numerical stability has not been performed for the general nonlinear equations, the stability criterion for linear hyperbolic equations (see, e.g., ref. 17) is apparently sufficient for the nonlinear equations. This is the well-known C-F-L (Courant, Friedrichs, Lewey) condition which essentially states that the domain of dependence of the calculated point must be included in the initial data. This means that the characteristic $C_1\ast$ through point $(i,2)$ in figure 6 must pass through or above point $(i-1,1)$. Similarly, the characteristic $C_2\ast$ through point $(i,J-1)$ must fall through or below point $(i-1,J)$.

Figure 6.---Step size limitation.

Strictly followed, the C-F-L condition would require testing every shock and body point to determine the maximum allowable step size, $\Delta \xi_m$, which would insure stability. However, experience indicates that the condition is not overly restrictive in the sense that steps slightly larger than $\Delta \xi_m$ do not usually cause a violent instability. Therefore, it is adequate to test only at the body and, for bodies with circular cross section, on the windward side where $\Delta \xi_m$ is likely to be smallest, due to the low Mach number. The step size is then taken slightly less than the maximum; a value of $\Delta \xi = 0.8 \Delta \xi_m$ works well in most problems.

Accuracy and computing time.—The accuracy of a numerical computation is usually estimated by comparing results obtained with various mesh spacings.
Since the exact solution is usually unknown, one can only compare with the results of the finest mesh and observe how the error decreases. If the truncation errors are second order in terms of the mesh size, then halving the mesh size should reduce the error by one-quarter.

To test the accuracy of the present method, calculations were performed from $x/R_n = 2$ to $x/R_n = 3$, using 3, 5, and 7 planes and 5, 10, and 15 points along each plane. In each case the same starting conditions were used at $x/R_n = 2$. The results of this study are shown in table II. Table II(a) presents the shock angles and the surface pressure on the $\phi = 90^\circ$ plane at $x/R_n = 3$. These results show that the method is of second-order accuracy. Also, it should be noted that the results with $k = 5$ and 7 agree very well for $J$ fixed (moving horizontally in the table). The good accuracy with only a few planes is attributed to the use of trigonometric analysis for the cross-derivatives.

The computing time naturally increases as the mesh is refined; this is illustrated in table II(b), which lists the total number of points computed, the total execute time, and the time per point in minutes. The calculation was performed on an IBM 7094 Model 1 in FORTRAN IV (version 13 IBJOB processor). A unit time of about $0.13 \times 10^{-2}$ min per point was obtained with the finest mesh. The unit time increases as the number of points decreases, probably because of fixed input/output times. The actual and unit times are almost doubled when one global iteration is made at each station.

Numerical Differentiation

Discussed next is the problem of evaluating cross-derivatives appearing in the equations. Partial derivatives in three directions appear in the functions $f_i^*$ defined by equations (46) through (49). They are of the form $\partial/\partial s^*$, $\partial/\partial \phi$, and $\partial/\partial \theta$, in the stream, radial, and circumferential directions. Special treatment is given to the circumferential derivative, following the discussion of the first two.

Streamwise and radial derivatives.- These derivatives are taken together since they are both evaluated by the standard polynomial approach. The main idea is to employ a differentiation formula consistent with the accuracy of the basic calculation.

For the streamwise derivative the approximation

$$\frac{\partial u^y}{\partial s^*} = \frac{(u^y)^i_j - (u^y)^i_{i-1}}{\Delta s^*}$$ \hspace{1cm} (64)

is clearly equivalent to the form of the difference equations employed. Equation (64) represents the derivative of $u^y$ at the midpoint of the interval, $\Delta s^*/2$, with an error of the order $\Delta s^*^2$. This derivative is evaluated at each step in the local iteration process previously described.
Because of the equally spaced coordinate mesh presently employed, the radial derivative is similarly determined to the same accuracy by the central difference

\[
\frac{\partial u^v}{\partial \eta} = \frac{(u^v)_{i+1} - (u^v)_{i-1}}{2 \Delta \eta}
\]  

(65)

Equation (65) approximates the derivative at point \((i,j)\) with an error of order \(\Delta \eta^2\). The average radial derivative at the midpoint \((i+1/2,j)\) between \(\xi_A\) and \(\xi_B\) can be obtained by means of the global iteration procedure previously described.

Standard end-point formulas of equivalent accuracy are used at the body and shock where central differences are not possible. These need not be written out, as they can be found in many books (e.g., ref. 18).

Circumferential derivatives.- In the aerodynamics of bodies it is well known, from linearized and perturbation theories as well as from experiment, that the solution of most problems can be represented by a trigonometric series in the meridional angle \(\phi\). When information such as this is available it should be possible, by choice of an appropriate functional form, to improve a numerical process. For example, with data known to have a nearly cosine variation, it is clear that fewer sample points are required to approximate the data with a cosine series than with a polynomial. A Fourier-series approximation is therefore used to evaluate derivatives with respect to \(\xi\). Symmetry conditions, which usually arise at \(\phi = 0\) and \(\phi = \pi\), are easily satisfied in this way. This technique makes it possible to calculate with fewer reference planes, thereby increasing the computational efficiency (see ref. 13).

For the present application it is necessary to determine Fourier approximations for pressure \(p\), flow angle \(\theta\), crossflow angle \(\phi\), and density \(\rho\) from their values on \(L\) planes \(\phi_k (k = 1, 2, \ldots, L)\) with \(\phi_1 = 0\) and \(\phi_L = \pi\). Symmetry conditions for three variables (represented by \(u^v_k (v = 1, 2, 4)\)) are satisfied by a cosine series of the form

\[
u_k^v = \sum_{n=0}^{L-1} a_n^v \cos n \phi_k \quad (v = 1, 2, 4)
\]  

(66)

For the crossflow angle \((v = 3)\), which is zero at \(\phi = 0\) and \(\phi = \pi\), a sine series

\[
\phi_k = \sum_{n=1}^{L-1} b_n \sin n \phi_k
\]  

(67)

is necessary.
When equally spaced meridional planes \( \phi_k = \pi (k - 1)/(L - 1) \), 
\((k = 1, 2, \ldots, L)\) are used, the calculation of the Fourier coefficients is particularly simple because the usual orthogonality conditions are exactly satisfied by a finite sum (see, e.g., ref. 18 or 19). In this case the coefficients are given by

\[
a_n = \frac{2}{L - 1} \left[ \frac{(u_1 + u_L \cos n\pi)}{2} \sum_{k=2}^{L-1} u_k \cos n\phi_k \right]
\]

and

\[
b_n = \frac{2}{L - 1} \sum_{k=2}^{L-1} \phi_k \sin n\phi_k \quad (n = 1, 2, \ldots, L - 1)
\]

Derivatives with respect to the angle \( \phi \) are obtained by differentiating the Fourier series, equations (66) and (67), to give

\[
\frac{d u_k}{d\phi} = \sum_{n=1}^{L-1} -n a_n \sin n\phi_k
\]

and

\[
\frac{d\phi_k}{d\phi} = \sum_{n=1}^{L-1} nb_n \cos n\phi_k
\]

The desired derivatives in terms of distance along the \( \zeta \) direction are obtained from equations (70) and (71) according to

\[
\frac{3}{\delta\zeta} = \frac{1}{g_{\zeta}} \left( \frac{d}{d\phi} \right)_{\text{on } \zeta}
\]

where it is understood that the derivative on the right side of (72) is evaluated using data on the \( \zeta \) coordinate. The scale factor \( g_{\zeta} \) relates the distance along \( \zeta \) corresponding to an incremental change in \( \phi \),

\[
g_{\zeta} = \frac{d\zeta}{d\phi}
\]

26
and can be written in terms of the direction cosines developed in appendix A. From these coordinate relations one has

$$r(d\phi/d\zeta) = \hat{e}_\phi \cdot \hat{\zeta} = \bar{v}_{33}$$

and therefore,

$$g_{\zeta} = (r/\bar{v}_{33})$$  \hspace{1cm} (73)

Starting Data

The method of characteristics, being a method for initial value problems, requires starting data which are usually obtained from boundary conditions or from an initial solution obtained by other methods. For present applications two types of starting conditions are of particular interest. These are (1) the spherically blunted body with the sonic line located on the spherical nose, and (2) pointed bodies which can be approximated by a cone in some small region near the tip.

**Sphere.**—Since a sphere does not have a preferred orientation, the flow remains symmetric around the wind axis for any angle of attack. Therefore, axisymmetric blunt-body solutions obtained, for example, by the inverse method of reference 20 can be used to provide initial conditions. It is necessary, however, to generate these axisymmetric starting data on an initial \( \eta-\zeta \) surface which is defined in a body axis system; the data will not be symmetric with respect to the body axis.

The characteristics program is set up to generate body axis data from wind axis data obtained from a blunt-body solution. Given these wind axis data on one body normal (see fig. 7(a)) where the flow is supersonic, say \( M > 1.05 \), the characteristics calculation is performed in a wind axis system and for \( \alpha = 0 \) to the position \( x_k' \), locating the body normals \( \eta_k' \). (Primes denote wind axes.) This is illustrated in figure 7(a). Since the flow is axisymmetric in terms of wind axes, the normals \( \eta_k' \) may be placed at an arbitrary circumferential position. The values of \( x_k' \) are chosen so that the normals \( \eta_k' \) match the ring of normals \( \eta_k \) emanating from the sphere at \( (x_0, r_0) \) in terms of body axes (fig. 7(b)). They are related to the meridional angle \( \phi_k \) and the angle of attack \( \alpha \) by

$$x_k' = R_n + (x_0 - R_n) \cos \alpha - r_0 \sin \alpha \cos \phi_k$$  \hspace{1cm} (74)

where \( R_n \) is the radius of the spherical nose.
Scalar data derived in this way on the body normals $n_k$ ($k = 1, 2, \ldots, L$) are ready for use in the general three-dimensional calculation. However, the flow angles, $\theta$ and $\phi$, must be recalculated in terms of body axes by means of the transformation relationships for velocity components. By the definition of $\theta$ and $\phi$ (fig. 1),

$$\theta = \sin^{-1} \frac{v}{V \cos \phi} \quad (75)$$

$$\phi = \sin^{-1} \frac{w}{V} \quad (76)$$

The magnitude of the velocity is unchanged by the coordinate rotation, so that

$$V = V' \quad (77)$$

From reference 15, the velocity components $v, w$ can be expressed as follows in terms of wind axis components:

$$v = (u' \sin \alpha + v' \cos \alpha \cos \phi') \cos \phi + v' \sin \phi' \sin \phi \quad (78)$$

$$w = v' \sin \phi' \cos \phi - (u' \sin \alpha + v' \cos \alpha \cos \phi') \sin \phi \quad (79)$$

where $\phi'$ is obtained from

$$\cot \phi \sin \phi' - \cos \alpha \cos \phi' - \frac{(x' - R_n)}{R_n} \sin \alpha = 0 \quad (80)$$

Equations (75) through (80) determine $\theta$ and $\phi$, relative to body-axis coordinates, from $u'$ and $v'$ calculated in the wind-axis system.

**Cone.**- Conical solutions are defined as those which are independent of $\xi$—that is, all derivatives with respect to $\xi$ vanish. Conical flows can be obtained in two ways: (1) by solving the boundary value problem for the reduced equations with $\partial \phi / \partial \xi = 0$ (see, e.g., ref. 21), or (2) by the asymptotic solution of three-dimensional initial value problem with a conical body (see ref. 11 or 22). The latter approach is presently taken since it fits the general computation scheme with little change.

Approximate initial data for a cone are specified and the calculation downstream is carried on until the conical flow condition is met to a specified accuracy. This approach has the disadvantage of being generally more time-consuming than the boundary value method, but it avoids many difficulties inherent in boundary value problems. The situation is quite similar to the calculation of the supersonic blunt-body problem by an asymptotic

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4It is clear physically that the conical condition must be obtained far downstream from the approximate starting condition.
unsteady calculation. Approximate starting conditions are obtained either from the perturbation solution for cones at small angle of attack (ref. 23) or from a previous three-dimensional solution for a different Mach number or cone angle.

Two modifications to the general computational procedure are required to obtain the correct conical flow solutions for circular cones. First, it was found necessary to account for the Ferri vortical layer (ref. 24) by setting the entropy at all body points, except at the leeward plane of symmetry, equal to the entropy at the windward shock point. The entropy at the leeward body station, which is a singular point, must equal the entropy at the leeward shock point.\(^5\) Thus the following conditions apply to circular cones:

\[
S_{1,k} = S_{j,L} \quad (k = 2, 3, \ldots, L)
\]

and

\[
S_{1,1} = S_{j,1}
\]

Secondly, as a result of the entropy singularity at the leeward body point, the circumferential density derivative there is also singular. Therefore, data from the singular point must be excluded in the numerical differentiation for \(\partial p/\partial \zeta\).

**Smoothing**

Under certain conditions where the density or crossflow angle develop large radial gradients, the numerical calculation for these quantities appears to be unstable. This situation can arise in the region of the vortical layer on pointed cones and in the so-called entropy layer over blunted cones. A stabilizing difference scheme similar to that known as the Lax, or Lax-Wendroff method (refs. 26 and 27) is therefore used in these cases.

Equations (59) and (60) are modified for this purpose according to the following difference approximation:

\[
(u^\nu)_j^i - (u^\nu)_j^{i-1} = \left\{ F_\nu + k \Delta \eta^3 \Delta \zeta \frac{\Delta \eta^2}{2 \eta_j} \left[ (u^\nu)_j^{i-1} - 2(u^\nu)_j^{i-1} + (u^\nu)_j^{i-1} \right] \right\} \Delta \xi
\]

where \(\nu = 3\) or \(4\) represents \(\phi\) or \(\rho\), respectively. The second term in the bracket is proportional to the second derivative of \(u^\nu\), so that the

\(^5\)Additional singular points arise in more general conical flows (see, e.g., ref. 24 or ref. 25).
difference equation (82) approximates the differential equation

\[
\frac{\partial u^\nu}{\partial s^\nu} = F^\nu + K \frac{\partial^2 u^\nu}{\partial \eta^2} \tag{83}
\]

where

\[
K = \frac{k \Delta \eta^3}{2n_j \Delta \xi} \tag{84}
\]

and \(\eta_j\) is the value of \(\eta\) at the shock \(\eta_j = (J - 1) \Delta \eta\).

The additional term in equation (83) represents a diffusion process and is the stabilizing term in the type of difference scheme given by equation (82). The arbitrary constant \(k\) in equation (84) is included to allow control of the diffusion term. If \(k = 0\), equation (83) reduces to equation (59) or equation (60). For \(k = \eta_j / \Delta \eta\), equation (83) is essentially the same as the Lax scheme (ref. 26).

It is known (this is demonstrated below) that the Lax difference scheme provides too much diffusion; therefore, it is desirable to choose \(k\) between 0 and \(\eta_j / \Delta \eta\) in order to provide numerical stability without undue loss of accuracy. In most applications \(k\) has been set equal to 1, and this condition is presently termed a second-order Lax smoothing because in this case

\[
K = \frac{\Delta \eta}{2n_j \Delta \xi} \Delta \eta^2 = 0(\Delta \eta^2) \tag{85a}
\]

Thus, the second-order difference scheme approaches the differential equation like \(\Delta \eta^2\) as the mesh is refined, whereas the Lax method approaches linearly in \(\Delta \eta\). For large Mach numbers, an order-of-magnitude analysis reveals that

\[
K = 0 \left( \frac{1}{M^2 j^2} \right) \tag{85b}
\]

which gives \(K\) a value of about 1/1000 for a typical mesh.

\textsuperscript{6}This argument is not strictly rigorous, as pointed out in reference 27, if the dissipative term is of the same order as the truncation error of the other terms. Nevertheless, the analogy is qualitatively correct, especially when the step size is not small. Also, when the integration is carried over large distances the truncation error might tend to be random, while the dissipative term is always additive.
The effect of this difference approximation is illustrated in figure 8 for a $15^\circ$ sphere-cone at zero angle of attack. The figure shows the density distribution in the shock layer at a station 40 nose radii downstream from the stagnation point.

Calculations by the present method are compared with the method of reference 28 which accurately determines the density distribution in axisymmetric flow by the use of a stream function. Present results with second-order smoothing are shown in figure 8 by the broken lines for 15 and 20 mesh points. It is seen that these curves differ from the more exact result only where the second derivative, $\frac{\partial^2 \rho}{\partial \eta^2}$, is large, and that the error decreases as the mesh is refined. Results obtained without smoothing and with 15 mesh points are shown by the symbols. The calculations appear to be unstable without smoothing.

It is emphasized that the need for smoothing arises only when the computational mesh is coarse relative to the details of the flow field. Experience indicates that the smoothing can be eliminated if the mesh is refined sufficiently. This, however, is not practical in three-dimensional problems where computation time and computer storage limitations are important factors. Therefore the proposed smoothing scheme provides a means for obtaining meaningful results with a relatively coarse mesh of points.

Discontinuous derivatives.- The possibility of discontinuities in supersonic flows presents a severe test for a numerical calculation. With the present numerical method, characteristic lines are not followed and some smearing of the discontinuities will occur because of the interpolation for data on...
the initial data line. This problem was therefore investigated and it was found that the proposed method does provide an accurate resolution of known flow details obtained with a standard method of characteristics for axisymmetric flow (ref. 29).

Continuous derivatives arise primarily on the body surface at points where the surface curvature is not analytic. As an example, figure 9 shows a map of the pressure in the vicinity of the sphere-cone juncture for a blunted cone at 5° angle of attack. Each line represents the pressure variation along a body normal, \( \eta \), between the body and the shock wave. The origin for each line is displaced in proportion to the \( x \) coordinate of the body point so that the circular symbols represent the actual surface-pressure distribution. The surface pressure decreases uniformly on the spherical nose until the sphere-cone juncture is reached, at \( x_b = 0.5 \), where the pressure gradient, \( \partial p/\partial \eta \), changes abruptly. The discontinuity from that point moves out into the flow field along a Mach wave. (The approximate position of the wave is indicated by the arrows.) The pressure is nearly constant behind the wave and varies according to the blunt-nose flow ahead of the wave. The curves theoretically should have a discontinuous slope there. As seen in the figure, the curves are only slightly rounded by the quadratic interpolation presently employed.

RESULTS AND DISCUSSION

Many numerical solutions have been obtained with the described method of characteristics, and these results compare favorably with other published works (see, ref. 13). Presented in this section are the details of a typical calculation for a 15° sphere-cone at 10° angle of attack and a Mach number of 10. This is considered to be sufficiently nonlinear to provide a good test of the theory. The dominant three-dimensional features of the flow are illustrated and compared with predictions of perturbation methods for small angles of attack. Bluntness effects are illustrated by comparison with a pointed cone solution, which is described first.
Pointed Cone

The solution for a pointed 15° cone at a Mach number of 10.6 and 10° angle of attack is presented in table III. Listed for each plane, \( \phi = \text{constant} \), are the shock angles, \( \sigma \) and \( \delta \). Below the shock angles are tabulated the \( x,r \) coordinates of each mesh point running from the shock to the body and the corresponding flow variables at those points, as labeled. The quantity \( M^* \) is the component of Mach number defined by the Mach angle \( \mu^* \) in equation (53). Remaining variables are defined in the list of symbols.

This conical solution was obtained with a coordinate mesh consisting of 9 meridional planes and with 11 points on each plane. Second-order smoothing was used on the density \( \rho \) and the crossflow angle \( \phi \) (smooth constant \( k = 1 \)). Initial data for the present case were obtained from a previous solution at \( M = 7 \) which agreed with the tabulated results of reference 11. The free-stream Mach number was changed from 7 to 10 and the computation was carried downstream until the flow relaxed to the new conical solution. This was accomplished in a number of stages, with each stage consisting of a computation from \( x = 0.8 \) to 1.0, using the output of the previous stage as initial data.

![Figure 10.- Relaxation of conical solution after N stages of calculation from x = 0.8 to x = 1.0; 15° cone, \( M_\infty = 10.6, \alpha = 10^\circ \).](image)

The accuracy of the solution is indicated by figure 10 which shows the shock angle in three meridional planes as a function of the reciprocal of the number of stages. It is seen that the approximate starting conditions cause the shock angles to change abruptly in the first stage with a slow decay to a limiting value as \( 1/N \) tends to zero. Conditions on the lee side are slowest to approach a limit. The shock angle at \( \phi = 0^\circ \) repeated to an accuracy of \( \Delta\sigma/\sigma = 0.45 \times 10^{-3} \) in the last stage of calculation and repeated to an accuracy of \( 1 \times 10^{-5} \) during the last step of the tenth stage.

The main features of this conical flow will be illustrated in conjunction with the blunted cone results described next.

Spherically Blunted Cone

The calculation for a spherically blunted cone with a 15° semivertex angle in air (\( \gamma = 1.4 \)) at \( M_\infty = 10 \) used 7 meridional planes with 15 points in each plane. Second-order smoothing, \( k = 1 \), was employed for the 10° angle of attack solution when \( x/R_n > 10 \); no smoothing was necessary for the other solutions presented.
The shock profiles in three meridional planes are shown in figure 11 for a cone length of 20 nose radii. Also shown on the figure are the pointed-cone shock positions which the blunt-cone shock must approach as \( x/R_n \) gets large. The small difference in the Mach number of the two solutions of 10 to 10.6 does not significantly affect the comparison. It is observed that the shock quickly approaches the pointed-cone shock for \( \phi = 90^\circ \) and \( \phi = 180^\circ \), while it does not appear to approach the pointed limit on the lee side, \( \phi = 0^\circ \). This is further illustrated in figure 12, which shows the circumferential shock shape for several axial stations. The slow decay on the lee side is evidenced by the hump in the profile near \( \phi = 0^\circ \). Note that the \( x \) coordinate is not constant on the shock traces shown, because the shock radial position is measured at its intersection with a cone normal to the body surface (see sketch in fig. 12).

The axial distribution of shock angle is shown in figure 13. On the windward plane, \( \phi = 180^\circ \), the angle reaches a minimum value of 13.744\(^\circ\) at \( x/R_n = 3.89 \). On the leeward plane the angle has not reached a minimum by \( x/R_n = 20 \) and is still well above the pointed-cone value at that point.

In a three-dimensional flow the streamlines generally flow across the meridional planes as a result of the asymmetrical pressure distribution. This crossflow is described in terms of the angle \( \phi \) which is related to the
familiar crossflow velocity according to $\phi = \sin^{-1} \left( \frac{w}{V} \right)$. Figure 14 shows the axial distribution of crossflow angle along the side meridian $\phi = 90^\circ$. The crossflow angle is a minimum at the sphere-cone juncture and then rises to a maximum value about 2.4 times the angle of attack. Thus, the surface upwash for blunted bodies can be greater than the maximum value of $2\alpha$ given by slender-body theory, which applies to pointed bodies at low supersonic speeds. It is interesting also to compare the result of the linearized characteristics method (ref. 15) which is in good agreement near the nose. Agreement extends to about $x/R_n = 10$, where the linear method starts to break down.

The reason for the breakdown is evident in figure 15 which shows the circumferential variation of $\phi$ for $\alpha = 10^\circ$ at various axial stations. Near the nose the variation is nearly sinusoidal, as assumed in the linearized method. However, the exact three-dimensional calculations show a large deviation from the sinusoidal variation beyond $x/R_n = 10$.

Figure 16 shows the variation of crossflow angle normal to the body. The curve for $x_b/R_n = 20$ approaches the pointed-cone solution near the shock, $n/n_s = 1$, but deviates by a large amount near the body. This is due to the low density of the flow in the entropy layer which is generated near the body surface by the blunt nose. The flow in this region has less momentum that that away from the surface, and is therefore turned more by the circumferential pressure gradient. The situation is analogous to boundary-layer
flow over an inclined body. Therefore, because of the entropy layer, the blunt-cone crossflow does not approach the pointed-cone distribution uniformly.

The entropy layer is characterized by a nearly constant pressure, as is shown in figure 17. On the other hand, the density varies strongly on the windward side of the body as may be seen in figure 18 for \( \frac{x_b}{R_n} = 16.7 \). The thickness of the entropy layer may be taken as the distance to the point where the density has a local maximum. This entropy layer is similar to that found in axisymmetric flow except that it develops faster (i.e., closer to the nose) on the windward side and more slowly on the lee side of the body.

Smoothing, which was used to stabilize the calculation, causes the peak in density to be rounded off. A theoretical maximum for \( \Phi = 180^\circ \) can be calculated by using the entropy corresponding to the minimum shock angle and by making use of the fact that the pressure is nearly constant. The theoretical maximum is about 15 percent higher than the calculated value. It is estimated that the smoothing had negligible effect on the density and crossflow angle for \( \eta/\eta_s \) greater than about 0.3 (see fig. 8).

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7Comparison is made in reference 14 between inviscid and experimental (viscous) surface streamlines.
Comparison With Experiment

Detailed experimental surveys of the flow field around blunted cones have been previously reported by Cleary (refs. 30 and 31). Experimental results from reference 30 are used to compare with the present theoretical results in order to confirm the validity of the proposed numerical methods. More complete comparisons between theory and experiment can be found in references 31 and 14, for both air and helium flows.

Figure 19 presents the surface pressure distribution along each of the 7 meridional planes along which the calculations were made. Theory and experiment are in excellent agreement although the experimental data tend to be slightly above the theory. This is consistent with the usual boundary-layer displacement effect. At $x/R_n = 20$ the blunt-cone pressure is very nearly equal to the pointed-cone value shown on the far right of figure 19.

Flow properties off the body surface are most easily studied experimentally by means of the impact or pitot pressure, as shown in figure 20. The pitot-pressure distribution normal to the body is presented for all 7 meridional planes and for an axial station of $x_p/R_n = 16.7$. The theory extends all the way to the shock in each case, while the data do not, except for $\phi = 60^\circ$ and $\phi = 150^\circ$ where experimental shock positions are indicated by a sharp drop in pressure.

In the viscous boundary layer the pitot pressure is low, going to zero at the body surface. Evidence of a very thick viscous layer is indicated by the experimental data for the lee side of the body, $\phi = 0$ and $30^\circ$. On the remaining meridional planes the boundary layer is very thin. The entropy layer is clearly indicated in the experimental data by the peak in the pitot pressure just off the windward side of the body. The theory underpredicts the peak value because of the previously described smoothing employed in the calculation. A theoretical maximum of $C_{p_p} = 9.03$ for $\phi = 180^\circ$ is determined by the use of the total pressure at the calculated minimum shock angle (fig. 13).
Aside from the noted differences due to the boundary layer and the entropy layer, the proposed numerical method appears to give an adequate prediction of shock-layer properties.

CONCLUDING REMARKS

Characteristics theory for three-dimensional steady flow was reviewed and the compatibility equations were derived in terms of pressure and stream angles as dependent variables. It was argued that the major practical difficulty encountered in bicharacteristic methods results from the need for numerical differentiation and interpolation of randomly spaced data. Furthermore, it was observed that shock-layer coordinates are essential to a simple treatment of circumferential derivatives on the shock surface. A reference plane method was therefore adopted in which an equal number of points are equally spaced between the body and shock along each reference plane. This mesh introduces the added complication of nonorthogonal coordinates into the equations but allows the use of simpler numerical techniques.

With the constraint of a uniformly spaced mesh, particular characteristic lines are not followed as in more standard characteristics methods. The role of characteristics theory in the reference plane method is to determine how the finite difference equations are to be locally solved. Since the characteristics are not traced throughout the shock layer, the possible coalescence of waves to form embedded shocks is not determined during the
calculation. However, characteristic lines can be traced afterward, from the numerical solution. Neglecting embedded shocks is not a serious deficiency since the calculations are correct for weak shocks (entropy rise being third-order in the deflection angle). In situations where strong shocks are suspected or known to occur (such as on flared bodies) special treatment would be required just as in standard two-dimensional methods.

Results for the three-dimensional flow around inclined blunted and pointed cones were presented to demonstrate the methods described and developed in this paper. Comparisons with a linearized characteristic method illustrated nonlinear angle-of-attack effects on crossflow parameters. Major effects of bluntness, which persist at large distances from the nose, were well predicted by the present methods. Predictions of surface and pitot pressures were found to be in good agreement with available experimental results for a 15° sphere-cone at 10° angle of attack.

While present examples were limited to bodies of revolution, the methods are not so restricted, being generally applicable to smooth bodies without axial symmetry. More complicated shapes typical of high-speed aircraft will require additional development, primarily in the area of special boundary conditions. For example, it should be possible, with special treatment of the leading-edge boundary condition, to calculate the flow over wings with supersonic leading edges. Finally, while there are no inherent limitations in angle of attack, it is clear that the flow near the leeward meridian must become wake-like at a sufficiently large inclination. In this case, it is essential to include the effects of viscosity and possible secondary shocks.

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APPENDIX A

DIRECTION COSINES FOR NONORTHOGONAL COORDINATES

In the derivation of the compatibility equations for the reference plane method it was necessary to transform from streamline coordinates to a non-orthogonal system consisting of \( s^* \), \( n \), and \( \zeta \). The direction \( s^* \) is the projection of the streamline on the meridional plane, \( n \) runs from body to shock in the meridional plane, and \( \zeta \) is the out-of-plane direction (see fig. 3). This transformation is expressed as

\[
\hat{x}_i = \varepsilon_{ij} \hat{z}_j^*
\]

where

\[
\hat{x}_i = (\hat{s}, \hat{n}, \hat{t})
\]

and

\[
\hat{z}_j^* = (\hat{s}^*, \hat{n}, \hat{\zeta})
\]

It is the purpose of this appendix to write out the expressions for the direction cosines \( \varepsilon_{ij} \). The analysis is made for the more general \( \xi, \eta, \zeta \) coordinates and is later specialized to the case where \( \xi = s^* \).

The directions of the nonorthogonal \( \xi, \eta, \zeta \) coordinates are determined step by step during the calculation as the shock shape is constructed (see section on Computational Procedure). When the location of the shock is known in terms of \( x, r, \phi \), the locations of field points between the shock and body are also determined by the prescribed spacing of the points along the body normals. The direction cosines can then be obtained by numerical differentiation as described below. This process is described for the shock wave but applies generally to each \( \zeta \) curve.

Let the shock surface be given by

\[
r_s = r_s(x, \phi)
\]

The intersection of the coordinate surface \( \xi = \text{constant} \) with the shock surface defines a curved line in space (a \( \zeta \) curve; see fig. 3). The \( x, r \) coordinates of this space curve depend on the meridional angle, \( \phi \), and on the body station, \( x_B \), which locates the \( \xi = \text{constant} \) surface. For each body station this dependence is denoted by

\[
x_s = \hat{x}_s(\phi)
\]

\[
r_s = \hat{r}_s(\phi)
\]
Equations (A3a) and (A3b) define two angles

\[
\tan \Delta x = \frac{1}{r_s} \frac{d\hat{x}_s}{d\phi}
\]

(A4)

\[
\tan \Delta r = \frac{1}{r_s} \frac{d\hat{r}_s}{d\phi}
\]

(A5)

which can be evaluated numerically. (The Fourier method previously described is used.) In figure 21 the unit vector \( \hat{\zeta} \) is related to \( \hat{e}_\phi \) through two rotations, the first rotation by \( \Delta x \) and the second by \( \tilde{\delta} \). It may also be verified from this figure that

\[
\tan \tilde{\delta} = \frac{dr}{r \, d\phi} \cos \Delta x
\]

and, by equation (A4),

\[
\tan \tilde{\delta} = \tan \Delta r \cos \Delta x
\]  

(A6)

Figure 21.- Coordinate vectors and angles.

The remaining two unit vectors are defined by angles \( \lambda \) and \( \sigma \); \( \hat{n} \) is rotated by \( \lambda \) from \( \hat{e}_r \) and \( \hat{\xi} \) is rotated by \( \sigma \) from \( \hat{e}_x \). Thus, one may write

\[
\hat{\xi} = \cos \sigma \hat{e}_x + \sin \sigma \hat{e}_r
\]  

(A7)

\[
\hat{n} = -\sin \lambda \hat{e}_x + \cos \lambda \hat{e}_r
\]  

(A8)

\[
\hat{\zeta} = \cos \tilde{\delta} \sin \Delta x \hat{e}_x + \sin \tilde{\delta} \hat{e}_r + \cos \tilde{\delta} \cos \Delta x \hat{e}_\phi
\]  

(A9)

The angle \( \lambda \), which determines the direction of \( \hat{n} \), may be arbitrarily specified. It is usually selected so that \( \hat{n} \) is normal to the body surface. The angles \( \sigma \) and \( \tilde{\delta} \) are determined at the shock by the calculated shock shape, and at the body by the given body shape. At intermediate points between the body and shock, \( \sigma \) and \( \tilde{\delta} \) can be similarly calculated from the \( x, r, \phi \) coordinates of the mesh points. These coordinates are known for any specified mesh spacing once the shock position is determined.

It must be noted, however, that \( \tilde{\delta} \) is distinct from the angle \( \delta \) which is used in the shock conditions developed in appendix B. From figure 21 it is found that

\[
\tan \delta = \frac{dr}{r \, d\phi} \tan \sigma
\]

\[
\text{or}
\]

\[
\tan \delta = \tan \Delta r \tan \sigma \tan \Delta x
\]

(A10)

When \( \hat{n} \) is chosen to be normal to the body axis, \( \Delta x = 0 \) and \( \delta = \tilde{\delta} \).
Having specified the directions of $\hat{\xi}, \hat{n}, \hat{\zeta}$ by equations (A7), (A8), and (A9), the task of determining $\varepsilon_{ij}^{*}$, appearing in equation (A1), can now be completed. Equations (A7), (A8), and (A9) are more conveniently written with index notation as

$$\hat{\xi}_i = -v_{ij} \hat{x}_j$$  \hspace{1cm} \text{(A11)}

where

$$v_{ij} = \begin{pmatrix} \cos \sigma & \sin \sigma & 0 \\ -\sin \lambda & \cos \lambda & 0 \\ \cos \delta \sin \Delta x & \sin \delta & \cos \delta \cos \Delta x \end{pmatrix}$$  \hspace{1cm} \text{(A12)}

The inverse transformation may be written

$$\hat{x}_i = v_{ij} \hat{z}_j$$  \hspace{1cm} \text{(A13)}

where

$$v_{ij} = (v_{ij})^{-1}$$

Although the matrix inversion can be performed numerically, for this problem it is more efficient to do the algebra beforehand, once and for all. This is done by taking scalar products of equation (A11) with $\hat{x}_j$ to obtain terms like $z_1 \cdot \hat{x}_1 = v_{11}$, $z_2 \cdot \hat{x}_2 = v_{12}$, and so on. On the other hand, the scalar products of equation (A13) with $\hat{z}_j$ give expressions such as

$$\hat{z}_1 \cdot \hat{x}_1 = v_{11} + v_{12} (\hat{\xi} \cdot \hat{n}) + v_{13} (\hat{\xi} \cdot \hat{\zeta})$$

$$\hat{z}_2 \cdot \hat{x}_1 = v_{11} (\hat{n} \cdot \hat{\xi}) + v_{12} + v_{13} (\hat{n} \cdot \hat{\zeta})$$

and so on. In this way, one obtains nine equations for the nine unknowns $v_{ij}$. The solution of this set of equations by Cramer's rule can be formally written as (see, e.g., ref. 18)

$$v_{ij} = \frac{v_{zj} f_{zj}}{D}$$  \hspace{1cm} \text{(A14)}

where $f_{zj}$ represents the cofactor matrix of the coefficients of $v_{ij}$.

$$f_{zj} = \begin{pmatrix} [1-(\hat{n} \cdot \hat{\xi})^2] & [(\hat{\xi} \cdot \hat{\zeta})(\hat{n} \cdot \hat{\xi})-(\hat{\xi} \cdot \hat{n})] & [(\hat{\xi} \cdot \hat{n})(\hat{n} \cdot \hat{\zeta})-(\hat{\xi} \cdot \hat{\zeta})] \\ f_{21} & [1-(\hat{\xi} \cdot \hat{\zeta})^2] & [(\hat{\xi} \cdot \hat{n})(\hat{\xi} \cdot \hat{\zeta})-(\hat{\xi} \cdot \hat{n})] \\ f_{31} & f_{32} & [1-(\hat{\xi} \cdot \hat{n})^2] \end{pmatrix}$$  \hspace{1cm} \text{(A15)}
where

\[ f_{\xi j} = f_{j\xi} \]

and the determinant \( D \) is given by

\[
D = 1 - (\hat{\eta} \cdot \hat{\zeta})^2 - (\hat{\xi} \cdot \hat{\eta})^2 - (\hat{\xi} \cdot \hat{\zeta})^2 - 2(\hat{\xi} \cdot \hat{\eta})(\hat{\xi} \cdot \hat{\zeta})(\hat{\eta} \cdot \hat{\zeta}) \quad (A16)
\]

Equation (A14) provides the direction cosines between \( \xi, \eta, \zeta \) and \( x, r, \phi \) coordinates. The corresponding relationships with \( s, n, t \) coordinates are obtained by use of equations (7), (8), and (9) which express the streamline directions in terms of \( \hat{e}_x, \hat{e}_r, \hat{e}_\phi \) and may be written

\[
\hat{x}_i = \tau_{ij} \hat{x}_j
\]

where

\[
\tau_{ij} = \begin{pmatrix}
\cos \phi \cos \theta & \cos \phi \sin \theta & \sin \phi \\
-\sin \theta & \cos \theta & 0 \\
-\sin \phi \cos \theta & -\sin \phi \sin \theta & \cos \theta
\end{pmatrix}
\]

(A18)

The substitution of equation (A13) into (A17) yields

\[
\hat{x}_i = \varepsilon_{ik} \hat{z}_k
\]

(A19)

where \( \varepsilon_{ik} \) is obtained from

\[
\varepsilon_{ik} = \tau_{ij} \nu_{jk}
\]

(A20)

Equation (A20) gives the direction cosines between \( \xi, \eta, \zeta \) and \( s, n, t \) coordinates. The explicit expressions for these direction cosines are involved, so the final combination of terms indicated by equation (A20) is left for the computer. The procedure is as follows. The matrix \( \nu_{ij} \), defined by (A12), is calculated with equations (A4), (A5), and (A6). The various scalar products in equations (A15) and (A16) are calculated from \( \bar{\nu}_{ij} \), and the matrix \( \nu_{ij} \) is then calculated from equation (A14). Finally, \( \varepsilon_{ik} \) is determined from equations (A18) and (A20).

The procedure described applies to general nonorthogonal coordinates \( \xi, \eta, \zeta \). The transformation used in equation (36) is a special case where the \( \xi \) direction is the streamline projection \( s^* \). In this case, equation (A9) is replaced by
\[ \hat{s}^* = \cos \theta \hat{e}_x + \sin \theta \hat{e}_r \]  
(A21)

Then the subsequent relations will be modified accordingly and there is obtained

\[ \varepsilon_{ik}^* = \tau_{ij} \nu_{jk}^* \]  
(A22)

which determines the desired transformation relationship indicated by equation (A1).
APPENDIX B

SHOCK-BOUNDARY CONDITIONS FOR THREE-DIMENSIONAL FLOW

Consider an elemental portion of the shock surface with an outer unit normal \( \hat{N} \), as shown in figure 22, and with unit vectors \( \hat{T} \) and \( \hat{L} \) completing an orthogonal set. The tangent vector \( \hat{T} \) can be chosen such that the \( N-T \) plane is parallel to the direction of the free-stream velocity. This choice will permit evaluation of the jump conditions in the \( N-T \) plane with two-dimensional shock relations. Let \( \hat{s}_\infty \) be a unit vector parallel to the free-stream velocity vector and with components

\[
\hat{s}_\infty = \cos \alpha \hat{e}_x + \sin \alpha \cos \phi \hat{e}_T - \sin \alpha \sin \phi \hat{e}_\phi
\]  

(B1)

in a cylindrical coordinate frame. The desired tangent vector \( \hat{T} \) can be constructed from \( \hat{s}_\infty \) and \( \hat{N} \) by means of two vector- or cross-products. The first product

\[
a \hat{L} = \hat{s}_\infty \times \hat{N}
\]  

(B2)

produces a vector parallel to \( \hat{L} \), and the second product

\[
\hat{T} = \hat{N} \times \hat{L}
\]  

(B3)

results in a vector which is normal to both \( \hat{N} \) and \( \hat{L} \), and which lies in the \( s_\infty-N \) plane. The factor \( a \) in equation (B2) is equal to the sine of the angle between \( \hat{s}_\infty \) and \( \hat{N} \) and may be evaluated from the scalar product

\[
b = \hat{s}_\infty \cdot \hat{N} = \cos(s_\infty, \hat{N})
\]  

(B4)

to give

\[
a = \sqrt{1 - b^2} = \sqrt{1 - (\hat{s}_\infty \cdot \hat{N})^2}
\]  

(B5)

Combining equations (B2) and (B3) and expanding the resulting vector triple product one obtains

\[
\hat{T} = \frac{1}{a} [\hat{N} \times (\hat{s}_\infty \times \hat{N})]
\]

or

\[
\hat{T} = \frac{1}{a} (\hat{s}_\infty - b\hat{N})
\]  

(B6)

45
In terms of components $T_x$, $T_r$, and $T_\phi$, equation (B6) becomes

$$\hat{T} = \left(\frac{s_x - b N_x}{a}\right) \hat{e}_x + \left(\frac{s_r - b N_r}{a}\right) \hat{e}_r + \left(\frac{s_{\phi} - b N_{\phi}}{a}\right) \hat{e}_{\phi}$$

Equation (B7) permits evaluation of the true inclination of the shock surface, and therefore allows the jump conditions to be determined with standard planar shock relations (see, e.g., ref. 32). The angle $\sigma'$ between $\hat{s}_\infty$ and $\hat{T}$ is given by

$$\cos \sigma' = \hat{s}_\infty \cdot \hat{T}$$

or

$$\cos \sigma' = s_x T_x + s_r T_r + s_{\phi} T_{\phi}$$

In the following development $\sigma'$ is expressed in terms of two angles measured in the cylindrical coordinates.

Let $\sigma$ be the angle between $\hat{e}_x$ and the trace of the shock surface on the plane $\phi = \text{constant}$, and let $\delta$ be the angle between $\hat{e}_\phi$ and the shock trace on the plane $x = \text{constant}$ as illustrated in figure 23. The shock angle $\delta$ is obtained by numerical differentiation as described in appendix A (see eq. (A10) and fig. 21). It is easily verified that the following relations hold between the angles shown in figure 23.

$$N_x = -N_m \sin \sigma$$

$$N_r = N_m \cos \sigma$$

$$N_{\phi} = -N_m \cos \sigma \tan \delta$$

where $N_m$ is the projection of $\hat{N}$ on the $x-r$ plane and satisfies the relation

$$N_m^2 + N_{\phi}^2 = 1$$

Substitution of $N_\phi$ from equation (B11) gives

$$N_m = \frac{1}{\sqrt{1 + \cos^2 \sigma \tan^2 \delta}}$$

Figure 23.- Shock angles.
and the shock normal vector may finally be written in terms of $\sigma$ and $\delta$ as

$$\hat{N} = \frac{(-\sin \sigma \hat{e}_x + \cos \sigma \hat{e}_r - \cos \sigma \tan \delta \hat{e}_\phi)}{\sqrt{1 + \cos^2 \sigma \tan^2 \delta}}$$

(B13)

The true shock angle can now be evaluated in terms of $\sigma$ and $\delta$ by equations (B1), (B7), (B8), and (B13), and the jump conditions can be calculated.

It is now necessary to determine the flow angles $\theta$ and $\phi$ measured relative to the meridional planes.

The streamline direction, measured in the N-T plane (fig. 24), is turned from the free stream by the angle $\theta_2'$. Since the streamline tangent $\hat{s}_2$ lies, by definition, in the N-T plane, one may write

$$\hat{s}_2 = -\sin(\sigma' - \theta_2')\hat{N} + \cos(\sigma' - \theta_2')\hat{T}$$

(B14)

Using equations (B7) and (B13), the vectors $\hat{N}$ and $\hat{T}$ can be written in terms of their components to give

$$\hat{s}_2 = (A\hat{e}_x + B\hat{e}_r)\hat{e}_x + (A\hat{e}_r + B\hat{e}_r)\hat{e}_r + (A\hat{e}_\phi + B\hat{e}_\phi)\hat{e}_\phi$$

(B15)

where

$$A = -\sin(\sigma' - \theta_2')$$

$$B = \cos(\sigma' - \theta_2')$$

Equation (7), on the other hand, gives $\hat{s}_2$ in terms of $\theta$ and $\phi$ as follows:

$$\hat{s}_2 = \cos \phi_2 \cos \theta_2 \hat{e}_x + \cos \phi_2 \sin \theta_2 \hat{e}_r + \sin \phi_2 \hat{e}_\phi$$

(B16)

Thus, by equating components of equations (B15) and (B16) one obtains finally

$$\tan \theta = \frac{-\sin(\sigma' - \theta_2')N_r + \cos(\sigma' - \theta_2')T_r}{-\sin(\sigma' - \theta_2')N_x + \cos(\sigma' - \theta_2')T_x}$$

(B17)

and

$$\sin \phi = -\sin(\sigma' - \theta_2')N_\phi + \cos(\sigma' - \theta_2')T_\phi$$

(B18)

If for given free-stream conditions, $p_\infty$, $\rho_\infty$, $V_\infty$, the standard planar shock conditions are written in the form (see ref. 32):
then the equations developed in this appendix allow the three-dimensional shock conditions to be functionally written as

\[
\begin{align*}
\rho &= \rho(\sigma') \\
\theta &= \theta(\sigma') \\
\end{align*}
\]

This is the form of the shock conditions employed in equation (62).

The overall shock calculation is performed in a straightforward iterative manner. This procedure is started with known field data, including \( \sigma \) and \( \delta \), on an initial data surface. A shock point on the new data surface is determined by the average value of \( \sigma \) between the initial and new shock points,

\[
\bar{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2)
\]

To begin, \( \sigma_2 \) is set equal to \( \sigma_1 \) and then subsequent estimates are made (either by the Newton method or by the bisecting method) until the pressure from equation (B20) agrees with the pressure calculated from the compatibility equation (57). During each iteration the current value of \( \sigma = \sigma_2 \) and the old value of \( \delta = \delta_1 \) are used to determine the true shock angle \( \sigma' \) by equation (B8). The pressure and other variables are then calculated from equations (B19). When this has been done for all the shock points on the new data surface, new values for \( \delta \) can be determined by numerical differentiation with respect to \( \Phi \) as described in appendix A. The entire process can then be repeated to refine the results; this is discussed in the section on Global Iteration.
APPENDIX C

SURFACE BOUNDARY CONDITIONS FOR BODIES WITHOUT AXIAL SYMMETRY

For noncircular bodies the surface boundary is complicated by the fact that the surface normal is not in the meridional plane. This means the boundary condition will involve both flow angles \( \theta \) and \( \phi \). (In terms of velocities, all three components, \( u, v, w \), enter into the conditions since none of the components are parallel to the body surface.) In this appendix the boundary condition for \( \theta \), which was given previously in equation (61a), is derived from the tangency condition on the velocity vector.

Let the equation of the body be given by

\[ g(x,r,\phi) = r - f(x,\phi) \]  \hspace{1cm} (C1)

The unit outer normal to the surface may be expressed

\[ \hat{N} = \frac{\nabla g}{|\nabla g|} \] \hspace{1cm} (C2)

where \( \nabla \) is the vector gradient operator. In terms of cylindrical coordinates, one obtains

\[ \hat{N} = \frac{- \frac{\partial f}{\partial x} \hat{e}_x + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{e}_\phi}{\sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{1}{r} \frac{\partial f}{\partial \phi} \right)^2}} \] \hspace{1cm} (C3)

The derivatives of \( f \) in equation (C3) are related to the surface inclination angles, figure 25, according to

\[ \tan \delta_x = \left( \frac{\partial f}{\partial x} \right)_\phi \] \hspace{1cm} (C4)

and

\[ \tan \delta_\phi = \frac{1}{r} \left( \frac{\partial f}{\partial \phi} \right)_x \] \hspace{1cm} (C5)

From equation (7) the streamline direction is expressed in terms of flow angles \( \theta \) and \( \phi \)

\[ \hat{s} = \cos \phi \cos \theta \hat{e}_x + \cos \phi \sin \theta \hat{e}_r + \sin \phi \hat{e}_\phi \] \hspace{1cm} (C6)
The tangency condition, $\mathbf{s} \cdot \mathbf{N} = 0$, may now be obtained from the scalar product of equations (C3) and (C6). This gives

$$N_x \cos \phi \cos \theta + N_r \cos \phi \sin \theta + N_\phi \sin \phi = 0 \quad \text{(C7)}$$

where $N_x$, $N_r$, $N_\phi$ are the components of $\mathbf{N}$ along $\mathbf{e}_x$, $\mathbf{e}_r$, $\mathbf{e}_\phi$, and are identified by equation (C3). Equation (C7) may be written as a quadratic relation in $\tan \theta$

$$(N_r^2 - N_\phi^2 \tan^2 \phi) \tan^2 \theta + 2N_x N_r \tan \theta + (N_x^2 - N_\phi^2 \tan^2 \phi) = 0 \quad \text{(C8)}$$

The solution is

$$\tan \theta = \frac{-N_x \pm \frac{N_\phi \tan \phi}{N_r} \sqrt{1 + \left(\frac{N_x}{N_r}\right)^2 - \left(\frac{N_\phi}{N_r}\right)^2 \tan^2 \phi}}{[1 - (N_\phi/N_r)^2 \tan^2 \phi]} \quad \text{(C9)}$$

Rewriting equation (C9) in terms of the surface inclination angles given by equations (C4) and (C5), one obtains

$$\tan \theta = \frac{(\tan \delta_x + \tan \delta_\phi \tan \phi \sqrt{1 + \tan^2 \delta_x - \tan^2 \delta_\phi \tan^2 \phi})}{(1 - \tan^2 \delta_\phi \tan^2 \phi)} \quad \text{(C10)}$$

where the positive root is chosen so that $\theta$ is decreased when $\phi < 0$ and $\delta_\phi > 0$.

Equations (C10), (C4), and (C5) determine the flow angle $\theta$ in terms of the crossflow angle $\phi$ and the body geometry. It is easily verified that for zero crossflow, $\phi = 0$, equation (C10) reduces to

$$\tan \theta = \tan \delta_x$$

which is the usual condition for circular bodies.
REFERENCES


TABLE I.- EFFECT OF GLOBAL ITERATION ON SHOCK ANGLE
[15° sphere-cone; M = 10, α = 10°, 15 points and 7 planes]

(a) Calculation from $x/R_n = 2.0$ to $x/R_n = 3.0$

<table>
<thead>
<tr>
<th>Plane, $\phi$, deg</th>
<th>Iterations</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\sigma$, deg at $x/R_n = 3.0$</td>
<td>$x/R_n = 3.0$</td>
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<tr>
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<td>29.0763</td>
<td>29.0763</td>
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<tr>
<td>30</td>
<td>27.6271</td>
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<td>60</td>
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<td>23.7137</td>
</tr>
<tr>
<td>90</td>
<td>18.8404</td>
<td>18.8417</td>
</tr>
<tr>
<td>120</td>
<td>15.5193</td>
<td>15.5194</td>
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<tr>
<td>150</td>
<td>14.2633</td>
<td>14.2633</td>
</tr>
<tr>
<td>180</td>
<td>13.9813</td>
<td>13.9813</td>
</tr>
</tbody>
</table>

(b) Calculation from $x/R_n = 10.0$ to $x/R_n = 11.0$

<table>
<thead>
<tr>
<th>Plane, $\phi$, deg</th>
<th>Iterations</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
</tr>
<tr>
<td></td>
<td>$\sigma$, deg at $x/R_n = 11.0$</td>
<td>$x/R_n = 11.0$</td>
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<tr>
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<td>180</td>
<td>17.4416</td>
<td>17.4415</td>
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</table>
TABLE II.- ACCURACY AND COMPUTING TIME

[15° sphere-cone; M = 10, α = 10°. Calculation from x/Rn = 2 to x/Rn = 3.]

(a) Shock angles and surface pressure on φ = 90° plane, x/Rn = 3

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<tr>
<th>Points</th>
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<th>K = 5</th>
<th>K = 7</th>
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(b) Computing time

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<tr>
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TABLE III. \textbf{-} 15° POINTED CONE SOLUTION

<table>
<thead>
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<th>P</th>
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<th>M</th>
<th>V</th>
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</thead>
<tbody>
<tr>
<td>0.9833</td>
<td>0.32995E-03</td>
<td>0.24896E-03</td>
<td>0.70028E-03</td>
<td>0.4783</td>
</tr>
<tr>
<td>0.9833</td>
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<tr>
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<td>0.4783</td>
</tr>
</tbody>
</table>

**PLANE 1** ANGL = 0.00 DEG LEeward PLAN

**FIELD DATA** SHOCK ANGLES, DEG SIGMA = 18.512 DEG DELTA = 0.0000

<table>
<thead>
<tr>
<th>X</th>
<th>P</th>
<th>RHO</th>
<th>M</th>
<th>V</th>
</tr>
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<tbody>
<tr>
<td>0.9833</td>
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<td>0.70028E-03</td>
<td>0.4783</td>
</tr>
</tbody>
</table>

**PLANE 2** ANGL = 22.50 DEG windward PLAN

**FIELD DATA** SHOCK ANGLES, DEG SIGMA = 18.512 DEG DELTA = 1.5917

<table>
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<th>P</th>
<th>RHO</th>
<th>M</th>
<th>V</th>
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<tbody>
<tr>
<td>0.9833</td>
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**PLANE 3** ANGL = 45.00 DEG leeward PLAN

**FIELD DATA** SHOCK ANGLES, DEG SIGMA = 18.575 DEG DELTA = 1.5278

<table>
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**PLANE 4** ANGL = 67.50 DEG windward PLAN

**FIELD DATA** SHOCK ANGLES, DEG SIGMA = 18.299 DEG DELTA = 1.5439

<table>
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**EQUIPEMENT** PLANE SHOCK ANGLES, DEG SIGMA = 18.512 DEG DELTA = 0.0000

<table>
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<th>RHO</th>
<th>M</th>
<th>V</th>
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<td>0.31556</td>
<td>0.24162</td>
<td>-5.18915E-08</td>
<td>0.12372</td>
</tr>
<tr>
<td>0.98756</td>
<td>0.31556</td>
<td>0.24162</td>
<td>-5.18915E-08</td>
<td>0.12372</td>
</tr>
</tbody>
</table>

**TABLE III. - 15° POINTED CONE SOLUTION - Concluded**

**PLANE A ANGL = 112.50 DEG**

**FIELD DATA**

<table>
<thead>
<tr>
<th align="center">SNOWANGLES, DEG</th>
<th align="center">SIGMA</th>
<th align="center">DELTA</th>
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<th align="center">X</th>
<th align="center">R</th>
<th align="center">PHIG</th>
<th align="center">PHII</th>
<th align="center">P</th>
<th align="center">RHOI</th>
<th align="center">M</th>
<th align="center">M*</th>
</tr>
</thead>
<tbody>
<tr>
<td align="center">0.98378</td>
<td align="center">0.31377</td>
<td align="center">0.22794</td>
<td align="center">-7.16727E-08</td>
<td align="center">0.17829</td>
<td align="center">0.7472E-04</td>
<td align="center">0.13654</td>
<td align="center">0.17213E-04</td>
<td align="center">0.5083E-04</td>
<td align="center">2.4744E-04</td>
<td align="center">2.4744E-04</td>
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</tbody>
</table>

**PLANE 7 ANGL = 135.00 DEG**

**FIELD DATA**

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**PLANE H ANGL = 157.50 DEG**

**FIELD DATA**

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**PLANE 9 ANGL = 180.00 DEG**

**FIELD DATA**

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**THETA HUB** 0.26179E-04