# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

**TECHNICAL NOTE 3224**

# THEORETICAL INVESTIGATION OF THE EFFECTS UPON LIFT OF A GAP BETWEEN WING AND BODY OF A SLENDER WING-BODY COMBINATION

By Duane W. Dugan and Katsumi Hikido

Ames Aeronautical Laboratory
Moffett Field, Calif.

NACA
WASHINGTON
August 1954

.N

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3224

# THEORETICAL INVESTIGATION OF THE EFFECTS UPON LIFT OF A GAP BETWEEN WING AND BODY OF A SLENDER WING-BODY COMBINATION

By Duane W. Dugan and Katsumi Hikido

## SUMMARY

Slender-body theory has been applied to the study of the effects upon lift produced by the presence of a gap between wing and cylindrical body of a slender wing-body combination. Two conditions were studied, one in which both wing and body had the same angle of attack, and the other in which only the deflected wing had an angle of incidence to the free stream. The lift for the case of combined angle of attack and wing deflection can be found by superposition.

The theory predicts large losses in lift even for minute gap widths; it is anticipated, however, that the effects of viscosity and possibly of compressibility not considered in the theory will serve to reduce such losses in practice in the case of very small gap widths. The loss in lift effectiveness due to gap effects is more severe when both wing and body are at an angle of attack than when only the wing has incidence relative to the free stream. For the wing-body combination exclusive of the nose and afterbody, the gap effects, expressed in terms of percent loss in lift, are more pronounced for larger ratios of body radius to wing semispan; and the ratio of the lift obtained from wing deflection to that due to angle of attack increases with increasing gap width. The effect of the lift of the nose is to reduce the percent loss in lift due to gap in the angle-of-attack case from that predicted for the portion of the configuration considered above. As a consequence, the effectiveness of the wing as a control surface may increase or decrease with increasing gap width accordingly as the ratio of body radius to wing semispan is small or large.

## INTRODUCTION

In connection with the use of all-movable lifting surfaces in missile design, there arises the practical problem of the effects upon the aerodynamic characteristics of the missile caused by the presence of a gap between the wing panels and the fuselage. When the fuselage is cylindrical, the gap is unavoidable for two reasons: In the first place,

a clearance between the movable wing panels and the body is required from mechanical considerations, and secondly, a space between the wing panels and the curved surface of the fuselage is created by the deflection of the wing with respect to the body. This latter gap distance varies in the chordwise direction. For the usual ratios of wing chord to body radius encountered in missile design, and for small angles of wing deflection, this variable portion of the gap width will be very small compared to the constant gap present at zero deflection. In the present theoretical treatment of the problem, only small angles of wing deflection are considered, and therefore the gap is assumed to have constant width.

The purpose of this report is to evaluate the effects of gap upon the lift of a typical slender wing-body combination having an all-movable wing. Two basic problems are considered: Problem one deals with the lifting characteristics of a wing-body combination inclined at a small angle of attack with respect to the free stream, the wing panels having zero deflection with respect to the body; problem two is concerned with the same configuration, but the body is considered to be at zero angle of attack and the wing is deflected to some small angle with reference to the body axis. By the principle of superposition, the lift of the wing-body combination having both angle of attack and wing deflection can be found.

Although the first of the problems described above could be treated by the method given in reference 1, where the effects of a gap in the middle part of a wing at subsonic speeds are investigated, the second problem appears less amenable to solution by such methods. More useful for the present purpose, it is believed, is the procedure employed in reference 2, in which slender-body theory is applied to the analysis of the aerodynamic characteristics of a family of wing-body combinations. Accordingly, both problems are treated in much the same manner as that given in the latter reference.

While the present report was in preparation, the results of other investigations of gap effects were published. Bleviss and Struble (ref. 3) suggested a method of estimating effects of streamwise gaps upon the lift of wing-body combinations which included replacing the body by an infinite wall. Mirels (ref. 4) obtained the slender-body result for problem one of the present analysis, but did not obtain in closed form the solution of the second problem.

In common with the above-mentioned investigations, the present analysis makes use of several simplifying assumptions. For example, the wing is considered to be vanishingly thin, and the fluid medium is assumed to have zero viscosity and to be free from shock waves. Consequently, the usual side-edge condition of infinite velocity and zero lift is applied at the edge of each wing panel adjacent to the gap. It is also assumed that the lift is zero across the gap. Only small angles of attack and wing deflection are permitted, and the plane of the wing

is considered to include the center line of the body. Finally, the wing-gap-body combination is regarded as being slender in the sense that lateral dimensions are small in comparison to longitudinal (free-stream) dimensions.

For gap widths of the order of a few boundary-layer thicknesses, the neglect of viscosity and compressibility will certainly produce unrealistic results. For example, subsonic theory (ref. 1) predicts large losses in lift for even infinitesimal gaps in the middle part of a wing. It is interesting to note, however, that in an early experiment treating the configuration of reference 1, Munk and Carlo (ref. 5) obtained results which indicate appreciable decreases in lift-curve slope at zero angle of attack for gap widths which, though not "infinitesimal," were small in terms of the wing chord. It remains for experimental investigation at supersonic speeds to define the range of gap dimensions and of angles of attack for which the present inviscid theory is reasonably valid.

## GENERAL ANALYSIS

If it is assumed that for slender bodies (including wings) the velocity gradient in the free-stream direction $\varphi_{xx}$ is small compared with the velocity gradients in the y and z directions, then the well-known Prandtl linearized differential equation for the perturbation velocity potential $\varphi$ of a compressible flow in three dimensions

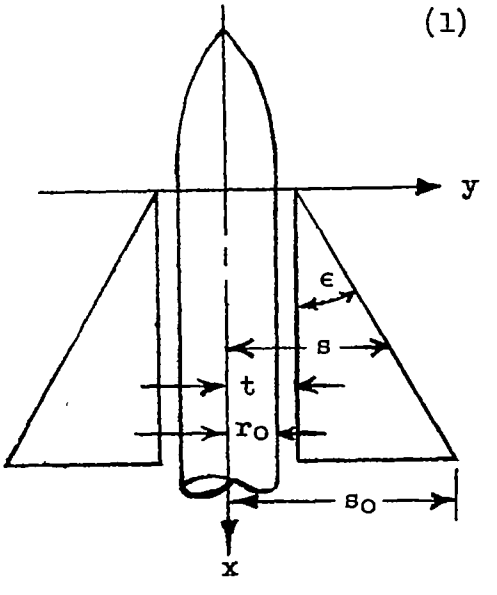
$$(1-M_o^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$$

is closely approximated by the more simple two-dimensional form

$$\varphi_{yy} + \varphi_{zz} = 0 \qquad (1)$$

which permits the use of conformal mapping. The same simplification can be made if $M_o$, the free-stream Mach number, is unity, or very close to unity, and the velocity gradient in the free-stream direction is not excessively large.

The wing-body combination chosen for consideration is shown in sketch (a). Symbols are defined in Appendix A. The body consists of a cylinder of circular cross section of constant radius and an arbitrarily pointed nose. The wing consists of two identical right triangular



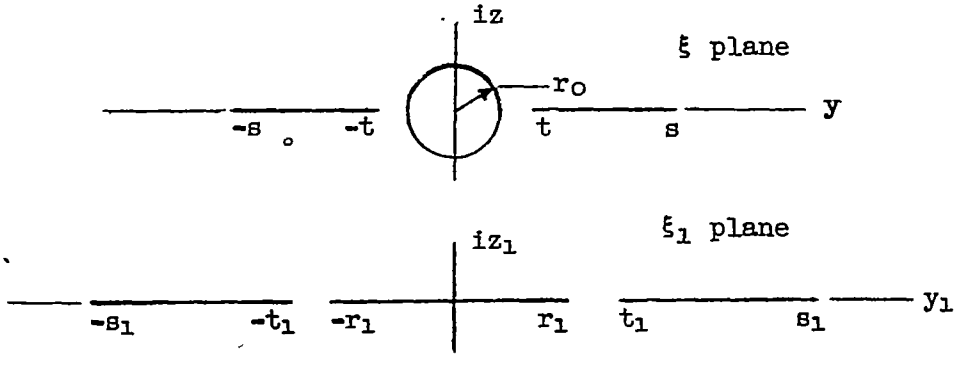

Sketch (a)

flat plates[1] of vanishing thickness placed adjacent to and on opposite sides of the constant cross-section portion of the body. The distance from the body axis to the inner edge of each wing panel is designated $t$; the gap is defined as the distance $t-r_o$, where $r_o$ is the radius of the cylinder. The distance from the body axis to the outermost point on the wing panel is called $s_o$; the symbol $s$ refers to the distance from the body axis to a point on the leading edge of each wing panel. The slope of the leading edge may be designated as $ds/dx$, or $m = \tan \epsilon$, where $\epsilon$ is the vertex angle of the wing panels. The origin of the coordinate system is taken at the point of intersection of the body axis and the line joining the apexes of the wing panels.

We introduce the complex variable $\xi = y + iz = \rho e^{i\theta}$. The cross section of the wing and body in the $\xi$ plane is shown in the upper part of sketch (b). The Joukowsky transformation

$$\xi_1 = \xi + \frac{r_o^2}{\xi}; \qquad \xi_1 = y_1 + iz_1 = \rho_1 e^{i\theta_1} \tag{2}$$

maps the given cross section onto the real axis of the $\xi_1$ plane, as shown in the lower part of sketch (b).




Sketch (b)

[1]The results of this analysis are, however, applicable to configurations in which the leading edge is not a straight line, provided, of course, that the assumptions of slender body theory are not violated.

Then, for $z_1 = 0$,

$$\left.\begin{aligned} y_1 &= y + \frac{r_o^2}{y} && y_1^2 \geq r_1^2 \\ y_1 &= 2r_o \cos\theta = 2y && y_1^2 \leq r_1^2 \\ s_1 &= s + \frac{r_o^2}{s} \\ t_1 &= t + \frac{r_o^2}{t} \\ r_1 &= 2r_o \end{aligned}\right\} \quad (3)$$

and

$$\left.\begin{aligned} y &= \frac{y_1 + \sqrt{y_1^2 - r_1^2}}{2} && y_1 \geq r_1 \\ y &= \frac{y_1 - \sqrt{y_1^2 - r_1^2}}{2} && y_1 \leq r_1 \\ y &= \frac{y_1}{2} && y_1^2 \leq r_1^2 \end{aligned}\right\} \quad (4)$$

The induced velocities in the two planes are related by

$$v - iw = (v_1 - iw_1)\frac{d\xi_1}{d\xi} \quad (5)$$

from which

$$\left.\begin{aligned} v &= v_1\left[1 - \left(\frac{r_o}{\rho}\right)^2 \cos 2\theta\right] + w_1\left(\frac{r_o}{\rho}\right)^2 \sin 2\theta \\ w &= w_1\left[1 - \left(\frac{r_o}{\rho}\right)^2 \cos 2\theta\right] - v_1\left(\frac{r_o}{\rho}\right)^2 \sin 2\theta \\ v_r &= \left(v_1 \cos\theta + w_1 \sin\theta\right)\left[1 - \left(\frac{r_o}{\rho}\right)^2\right] + 2w_1\left(\frac{r_o}{\rho}\right)^2 \sin\theta \end{aligned}\right\} \quad (6)$$

Also, Laplace's equation must be satisfied in the $\xi_1$ plane, so that

$$\varphi_{y_1 y_1} + \varphi_{z_1 z_1} = 0 \quad (7)$$

The general solution to equation (7) that gives the vertical induced velocity $w_1$ on the $y_1$ axis due to the jump in the value of the lateral velocity $\Delta v_1$ across the $y_1$ axis can be written

$$w_1(y_1) = -\frac{1}{2\pi}\int_{-s_1}^{s_1}\frac{\Delta v_1(y_2)}{y_1 - y_2}\,dy_2 \tag{8}$$

In the present problems, $w_1(y_1)$ is known from boundary conditions, but $\Delta v_1$ is unknown (except in the gap). Equation (8) is thus an integral equation which will be solved to find $\Delta v_1$ in each of the two problems.

## PROBLEM 1 - WING-BODY COMBINATION AT SMALL ANGLE OF ATTACK, WING AT ZERO ANGLE OF DEFLECTION

### Boundary Conditions

As was mentioned in the introduction, it is assumed that no lift is carried in the gap; hence the circulation and vorticity in the gap are taken to be zero. The boundary conditions which obtain in this problem are therefore

$$\left.\begin{array}{lll}
\text{(i)} & w = -V_o\alpha & \rho = r_o,\ 0 \le \theta \le 2\pi \\
\text{(ii)} & \Delta v = 0 & z = 0,\ r_o^2 < y^2 < t^2 \\
\text{(iii)} & w = -V_o\alpha & z = 0,\ t^2 < y^2 < s^2 \\
\text{(iv)} & v = 0,\ w = 0 & \rho = \infty,\ 0 \le \theta \le 2\pi \\
\text{(v)} & \Delta\varphi = 0 & z = 0,\ y^2 \ge s^2
\end{array}\right\} \tag{9}$$

In the $\xi_1$ plane these boundary conditions become

$$\left.\begin{array}{lll}
\text{(i)} & w_1 = -\dfrac{V_o\alpha}{2} & z_1 = 0,\ 0 \le y_1^2 \le r_1^2 \\
\text{(ii)} & \Delta v_1 = 0 & z_1 = 0,\ r_1^2 < y_1^2 < t_1^2 \\
\text{(iii)} & w_1 = -\dfrac{V_o\alpha}{2}\left(1 + \dfrac{y_1}{\sqrt{y_1^2 - r_1^2}}\right) & z_1 = 0,\ t_1^2 < y_1^2 < s_1^2 \\
\text{(iv)} & v_1 = 0,\ w_1 = 0 & \rho_1 = \infty,\ 0 \le \theta_1 \le 2\pi \\
\text{(v)} & \Delta\varphi_1 = 0 & z_1 = 0,\ y_1^2 \ge s_1^2
\end{array}\right\} \tag{10}$$

## Solution of the Integral Equation

The general solution expressed by equation (8) can be written

$$w_1(y_1) = -\frac{1}{2\pi}\int_{-s_1}^{-t_1} \frac{\Delta v_{1W}}{y_1 - y_2}\,dy_2 - \frac{1}{2\pi}\int_{-t_1}^{-r_1} \frac{\Delta v_{1g}}{y_1 - y_2}\,dy_2 -$$

$$\frac{1}{2\pi}\int_{-r_1}^{r_1} \frac{\Delta v_{1B}}{y_1 - y_2}\,dy_2 - \frac{1}{2\pi}\int_{r_1}^{t_1} \frac{\Delta v_{1g}}{y_1 - y_2}\,dy_2 - \frac{1}{2\pi}\int_{t_1}^{s_1} \frac{\Delta v_{1W}}{y_1 - y_2}\,dy_2 \tag{11}$$

where

$$\Delta v_{1W} = \Delta v_1 \text{ between } \pm t_1 \text{ and } \pm s_1$$

$$\Delta v_{1g} = \Delta v_1 \text{ between } \pm r_1 \text{ and } \pm t_1$$

$$\Delta v_{1B} = \Delta v_1 \text{ between } -r_1 \text{ and } r_1$$

Since $\Delta v_{1g} = 0$, the second and fourth integrals of equation (11) vanish. Also, since the wing-body combination is laterally symmetric,

$$\Delta v_1(y_1) = -\Delta v_1(-y_1) \tag{12}$$

and equation (11) can be written

$$w_1(\eta_1) = -\frac{1}{2\pi}\int_0^{r_1^2} \frac{\Delta v_{1B}}{\eta_1 - \eta_2}\,d\eta_2 - \frac{1}{2\pi}\int_{t_1^2}^{s_1^2} \frac{\Delta v_{1W}}{\eta_1 - \eta_2}\,d\eta_2 \tag{13}$$

where

$$\eta_1 = y_1^2$$

$$\eta_2 = y_2^2$$

Inversion of equation (13) will be accomplished by the application of the procedure presented in reference 2. First consider the region $0 \le \eta_1 \le r_1^2$. Here $w_1 = -\frac{V_0\alpha}{2}$ and equation (13) may be written

$$-\frac{V_0\alpha}{2} + \frac{1}{2\pi}\int_{t_1^2}^{s_1^2} \frac{\Delta v_{1W}(\eta_2)}{\eta_1 - \eta_2}\,d\eta_2 = -\frac{1}{2\pi}\int_0^{r_1^2} \frac{\Delta v_{1B}(\eta_2)}{\eta_1 - \eta_2}\,d\eta_2;\quad 0 \le \eta_1 \le r_1^2 \tag{14}$$

Equation (14) is now of the general form

$$f(\eta_1) = -\frac{1}{2\pi}\int_a^b \frac{\Delta v_1(\eta_2)}{\eta_1 - \eta_2}\, d\eta_2 \tag{15}$$

of which the inversion is

$$\Delta v_1(\lambda) = \frac{2}{\pi\sqrt{(b-\lambda)(\lambda-a)}}\left[\frac{1}{2}\int_a^b \Delta v_1(\eta_2)d\eta_2 + \int_a^b \frac{f(\eta_1)\sqrt{(b-\eta_1)(\eta_1-a)}}{\lambda - \eta_1}\, d\eta_1\right] \tag{16}$$

which for $\Delta v_1(a) = 0$ reduces to

$$\Delta v_1(\lambda) = \frac{2}{\pi}\sqrt{\frac{\lambda-a}{b-\lambda}}\int_a^b \frac{f(\eta_1)}{\lambda - \eta_1}\, d\eta_1 \sqrt{\frac{b-\eta_1}{\eta_1-a}} \tag{17}$$

In the present problem, $\Delta v_1(0) = 0$ from symmetry, and the inversion of equation (14) is

$$\Delta v_{1B}(\eta_2) = -V_0\alpha\sqrt{\frac{\eta_2}{r_1^2-\eta_2}} + \frac{1}{\pi}\sqrt{\frac{\eta_2}{r_1^2-\eta_2}}\int_{t_1^2}^{s_1^2} \frac{\Delta v_{1W}(\eta_3)}{\eta_2 - \eta_3}\sqrt{\frac{\eta_3-r_1^2}{\eta_3}}\, d\eta_3 \tag{18}$$

This expression for $\Delta v_{1B}$ will be used subsequently in solving for the lifting properties of the body. To find $\Delta v_{1W}$ it is necessary first to substitute equation (18) into equation (13) and reverse the order of integration. Using $w_1(\eta_1) = -\frac{V_0\alpha}{2}\left(1 + \sqrt{\frac{\eta_1}{\eta_1-r_1^2}}\right)$ for $t_1^2 < \eta_1 < s_1^2$ we obtain, after simplification,

$$-V_0\alpha = -\frac{1}{2\pi}\int_{t_1^2}^{s_1^2} \frac{\Delta v_{1W}(\eta_2)}{\eta_1 - \eta_2}\sqrt{\frac{\eta_2-r_1^2}{\eta_2}}\, d\eta_2 \tag{19}$$

Inversion of equation (19) will follow the procedure given by equations (15) and (16). Thus

$$\Delta v_{1W} = \frac{2\sqrt{\eta_1}}{\pi\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}} \left[ \frac{1}{2} \int_{t_1^2}^{s_1^2} \Delta v_{1W}(\eta_2) \sqrt{\frac{\eta_2-r_1^2}{\eta_2}}\, d\eta_2 - V_0\alpha \int_{t_1^2}^{s_1^2} \frac{\sqrt{(s_1^2-\eta_2)(\eta_2-t_1^2)}}{\eta_1 - \eta_2}\, d\eta_2 \right] \tag{20}$$

or

$$\Delta v_{1W} = \frac{2\sqrt{\eta_1}}{\pi\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}} \left\{ G - V_0\alpha \frac{\pi}{2} \left[ 2\eta_1 - (s_1^2+t_1^2) \right] \right\} \tag{21}$$

where

$$G = \frac{1}{2} \int_{t_1^2}^{s_1^2} \Delta v_{1W}(\eta_2) \sqrt{\frac{\eta_2-r_1^2}{\eta_2}}\, d\eta_2$$

Equation (21) is not unique since it contains an integral involving $\Delta v_{1W}$. However, the integral is merely a constant, which can be evaluated from the assumption that $\Delta\varphi_1$ is zero at $y_1^2 = t_1^2$. Knowing $\Delta v_1$, we can express $\Delta\varphi_1$ as

$$\Delta\varphi_1 = \int_{s_1}^{y_1} \Delta v_1 dy_1 \tag{22}$$

or, in terms of $\eta_1 = y_1^2$

$$\Delta\varphi_1 = \int_{s_1^2}^{\eta_1} \frac{\Delta v_{1W}}{2\sqrt{\eta_1}}\, d\eta_1 \tag{23}$$

Then

$$\Delta\varphi_1(t_1) = 0 = \int_{s_1^2}^{t_1^2} \frac{\Delta v_{1W}}{2\sqrt{\eta_1}}\, d\eta_1 \tag{24}$$

Substituting equation (21) into equation (24) yields

$$G\int_{s_1^2}^{t_1^2} \frac{d\eta_1}{\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}} = \frac{\pi}{2} V_o\alpha \int_{s_1^2}^{t_1^2} \frac{2\eta_1 - s_1^2 - t_1^2}{\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}} d\eta_1 \tag{25}$$

Performing the indicated integrations[2] and solving for G gives

$$G = \pi V_o\alpha \left[(s_1^2-r_1^2)\frac{E(k)}{K(k)} - \frac{1}{2}(s_1^2+t_1^2-2r_1^2)\right] \tag{26}$$

where

$$k = \sqrt{\frac{s_1^2-t_1^2}{s_1^2-r_1^2}}$$

An expression for $\Delta v_{1W}$ can now be found by substituting equation (26) into equation (21). The result is

$$\Delta v_{1W} = \frac{2\sqrt{\eta_1}\, V_o\alpha}{\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}} \left[(s_1^2-r_1^2)\frac{E(k)}{K(k)} - (\eta_1-r_1^2)\right] \tag{27}$$

## Span Loading on Wing Panel

The span loading is proportional to the total circulation $\Gamma = \Delta\Phi_{TE} - \Delta\Phi_{LE}$. From equations (23) and (27) we obtain

$$\Delta\Phi_{1W} = 2V_o\alpha\sqrt{s_1^2-r_1^2}\, Z(\psi_1,k) \tag{28}$$

where

$$Z(\psi_1,k) = E(\psi_1,k) - \frac{E(k)}{K(k)} F(\psi_1,k)$$

and

$$\psi_1 = \sin^{-1}\sqrt{\frac{s_1^2-y_1^2}{s_1^2-t_1^2}}$$

(The function Z is tabulated in ref. 7.)

[2]The elliptic integral of the third kind encountered here, as well as those which occur subsequently, were evaluated by use of reference 6.

If $t_1$ is set equal to $r_1$ (condition for a slender wing on an infinite cylinder with no gap) equation (33) becomes

$$(\Delta\varphi_{1W})_{t_1=r_1} = 2V_O\alpha\sqrt{s_1^2-y_1^2}$$

which when transformed by equations (3) may be written in the yz plane

$$(\Delta\varphi_{1W})_{t=r_O} = 2V_O\alpha\sqrt{s^2\left(1+\frac{r_O^4}{s^4}\right)-y^2\left(1+\frac{r_O^4}{y^4}\right)} \tag{29}$$

Equation (29) agrees with the expression for the velocity potential for a slender wing on a semi-infinite cylinder given in reference 8.

At the leading edge of the wing, $y_1 = s_1$, and $(\Delta\varphi_{1W})_{LE} = 0$ as it should. Therefore, $\Gamma_W$ can be given by equation (28) with $s_O$ substituted for $s$. However, it is more convenient to express the value of the circulation in terms of the value of the semispan of the wing for zero gap $s_O^*$. Then, since $s_O = s_O^* + g$, where $g$ is the gap distance, the expression for span loading is

$$\left(\frac{dL}{dy}\right)_W = \rho V_O\Gamma_W = 4qs_O^*\alpha\left(1+\frac{g}{s_O^*}\right)\left[1-\left(\frac{r_O}{s_O}\right)^2\right]Z(\psi_1,k) \tag{30}$$

where $\psi_1$ and $k$ are evaluated for $s_1 = s_{O1}$.

## Pressure Distribution Over Wing Panel

The loading coefficient $\Delta p/q$ can be found from the equation

$$\frac{\Delta p}{q} = 2\frac{\Delta u}{V_O} = \frac{2}{V_O}\frac{\partial\Delta\varphi}{\partial x} = \frac{2}{V_O}\frac{\partial\Delta\varphi}{\partial s_1}\frac{ds_1}{ds}\frac{ds}{dx}$$

First, the differentiation of $\Delta\varphi_{1W}$ with respect to $s_1$ is carried out and gives, after simplification,

$$\frac{\partial\Delta\varphi_{1W}}{\partial s_1} = 2V_O\alpha\frac{s_1\sqrt{s_1^2-r_1^2}}{s_1^2-t_1^2}\left[1-\frac{E(k)}{K(k)}\right]\left[Z(\psi_1,k)+\sqrt{\frac{(y_1^2-t_1^2)(y_1^2-r_1^2)}{(s_1^2-y_1^2)(s_1^2-r_1^2)}}\right] \tag{31}$$

Since $ds_1/ds = 1 - r_o^2/s^2$, and for triangular wings $ds/dx = m$, there results for the loading coefficient for the wing

$$\left(\frac{\Delta p}{q\alpha}\right)_W = \frac{4mt^2(s^4-r_o^4)}{(s^2-t^2)(s^2t^2-r_o^4)}\left[1 - \frac{E(k)}{K(k)}\right]\left[\left(1 - \frac{r_o^2}{s^2}\right) Z(\psi_1,k) + \frac{y^2 - r_o^2}{yt}\sqrt{\frac{(y^2-t^2)(y^2t^2-r_o^4)}{(s^2-y^2)(s^2y^2-r_o^4)}}\right] \tag{32}$$

In the case of zero gap ($t = r_o$), equation (32) reduces to the result given in reference 8 for the loading coefficient for a triangular wing on a semi-infinite cylindrical body.

## Lift of Wing Panels

The lift of the wing panels may be found by integrating the expression for the span loading; that is,

$$\left(\frac{L}{q}\right)_W = \frac{2}{V_o}\int_{-s_o}^{s_o} \Gamma_W dy = \frac{4}{V_o}\int_t^{s_o} (\Delta\varphi_W)_{TE} dy \tag{33}$$

The integration is more easily carried out in terms of $y_1$. Equation (33) then becomes

$$\left(\frac{L}{q}\right)_W = \frac{2}{V_o}\int_{t_1}^{s_{o1}} (\Delta\varphi_{1W})_{TE}\left(1 + \frac{y_1}{\sqrt{y_1^2-r_1^2}}\right) dy_1 \tag{34}$$

Substituting in equation (34) the expression for $(\Delta\varphi_{1W})_{TE}$ obtained from equation (28) with $s_1 = s_{o1}$ gives

$$\left(\frac{L}{q\alpha}\right)_W = \pi\left\{t_1^2 + (s_{o1}^2-r_1^2)\left[1 - 2\frac{E(k)}{K(k)}\right]\right\}\left[1 + \Lambda_o(\psi,k_1)\right] - \pi r_1^2 + 2t_1\sqrt{s_{o1}^2-r_1^2}\left\{E(k_1) - K(k_1) + \frac{r_1^2}{t_1^2}\left[1 - 2\frac{E(k)}{K(k)}\right]K(k_1)\right\} \tag{35}$$

where

$$\Lambda_0(\psi,k_1) = \frac{2}{\pi}\left\{\left[E(k_1) - K(k_1)\right]F(\psi,k_1') + KE(\psi,k_1')\right\}$$

$$\psi = \sin^{-1}\sqrt{\frac{s_1^2-r_1^2}{s_1^2}};\quad k_1 = \frac{r_1}{t_1}\,k = \frac{r_1}{t_1}\sqrt{\frac{s_1^2-t_1^2}{s_1^2-r_1^2}}$$

$$k_1' = \sqrt{1 - k_1^2}$$

and the moduli and arguments are evaluated at the wing trailing edge where $s_1 = s_{o1}$. (The function $\Lambda_0$ is tabulated in refs. 6 and 9.)

Equation (35) gives the lift of two pointed low-aspect-ratio wings in the presence of a slender body which is cylindrical in the region between the apexes and trailing edges of the wing panels and lies between and at a distance $t - r_o$ from the panels, all at an angle of attack.

If $t_1 = r_1 = 0$, equation (35) reduces to $(L/q\alpha)_W = 2\pi s_o^2$ or, for the two triangular wing panels now joined (triangular wing)

$$C_L = \frac{2\pi s_o^2}{S}\,\alpha = \frac{\pi}{2}\,A\alpha \tag{36}$$

which is the well-known result for the lift of a slender wing.

## Span Loading on Body

Equation (18) expresses the vorticity on the body in terms of the vorticity on the wing. Substituting equation (27) into equation (18) gives

$$\Delta v_{1B}(\eta_1) = -2V_o\alpha\sqrt{\frac{\eta_1}{r_1^2-\eta_1}}\left[\frac{1}{2} + \frac{1}{\pi}\int_{t_1^2}^{s_1^2}\frac{(\eta_3-r_1^2)d\eta_3}{(\eta_1-\eta_3)\sqrt{(s_1^2-\eta_3)(\eta_3-t_1^2)}} - \right.$$

$$\left.\frac{1}{\pi}(s_1^2-r_1^2)\,\frac{E(k)}{K(k)}\int_{t_1^2}^{s_1^2}\frac{d\eta_3}{(\eta_1-\eta_3)\sqrt{(s_1^2-\eta_3)(\eta_3-t_1^2)}}\right] \tag{37}$$

which becomes

$$\Delta v_{1B}(\eta_1) = 2V_0\alpha \sqrt{\frac{\eta_1}{r_1^2-\eta_1}} \left[ \frac{1}{2} - \frac{(r_1^2-\eta_1) + (s_1^2-r_1^2)\frac{E(k)}{K(k)}}{\sqrt{(s_1^2-\eta_1)(t_1^2-\eta_1)}} \right] \quad (38)$$

Since $\Delta\varphi_1 = 0$ in the gap, the jump in potential across the real axis between $-r_1$ and $r_1$ can be given by

$$\Delta\varphi_{1B} = \int_{r_1^2}^{y_1^2} \frac{\Delta v_{1B}}{2\sqrt{\eta_1}}\, d\eta_1 \quad (39)$$

From equations (38) and (39), it follows that

$$\Delta\varphi_{1B} = 2V_0\alpha \left[ \sqrt{s_1^2-y_1^2}\, \sin\psi_2 - \sqrt{s_1^2-r_1^2}\, Z(\psi_2,k) - \frac{1}{2}\sqrt{r_1^2-y_1^2} \right] \quad (40)$$

where

$$\psi_2 = \sin^{-1}\sqrt{\frac{r_1^2-y_1^2}{t_1^2-y_1^2}}$$

When $t_1 = r_1$, equation (40) reduces to

$$(\Delta\varphi_{1B})_{t_1=r_1} = 2V_0\alpha \left[ \sqrt{s_1^2-y_1^2} - \frac{1}{2}\sqrt{r_1^2-y_1^2} \right]$$

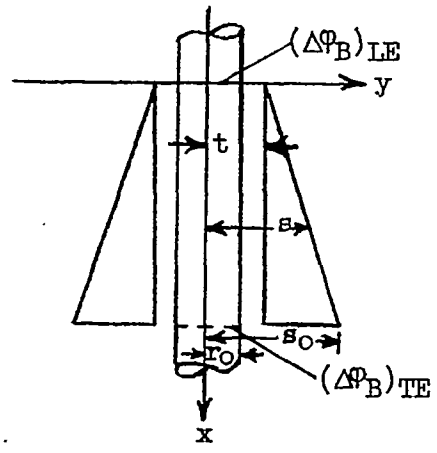
or, in physical terms, to

$$(\Delta\varphi_B)_{t=r_0} = 2V_0\alpha \left[ \sqrt{s^2\left(1+\frac{r_0^2}{s^2}\right)^2 - 4y^2} - \sqrt{r_0^2-y^2} \right]$$

The span loading on the body due to the presence of the lifting wing can now be determined. First we shall consider the loading induced on the body aft of the apex of the wing (see sketch (c)). The value of $(\Delta\varphi_{1B})_{LE}$ is found by setting $s_1 = t_1$ in equation (40), which becomes

$$(\Delta\varphi_{1B})_{LE} = V_o\alpha\sqrt{r_1^2-y_1^2}$$

For $\Delta\varphi_{1B}$ at the trailing edge, the value of $s_{o1}$ is used in place of $s_1$ in equation (40). Therefore, the span loading on the body is




Sketch (c)

$$\frac{dL}{dy_1} = 4q\alpha\left[\sqrt{s_{o1}^2-y_1^2}\sin\psi_2 - \sqrt{s_{o1}^2-r_1^2}\,Z(\psi_2,k) - \sqrt{r_1^2-y_1^2}\right] \quad (41)$$

or since the value of the span loading on the nose in $4q\alpha\sqrt{r_o^2-y^2}$ according to Munk's airship theory, the total span loading on the body is, in terms of the geometric properties,

$$\left(\frac{\frac{dL}{dy}}{4\alpha q s_o^*}\right)_B = \left[1+\frac{g}{s_o^*}\right]\left[\sqrt{\left(1+\frac{r_o^2}{s_o^2}\right)^2 - 4\frac{y^2}{s_o^2}}\sin\psi_2 - \left(1-\frac{r_o^2}{s_o^2}\right)Z(\psi_2,k) - \sqrt{\left(\frac{r_o}{s_o}\right)^2-\left(\frac{y}{s_o}\right)^2}\right] \quad (42)$$

where $\psi_2$ and k are evaluated using $s_1 = s_{o1}$.

From equations (28), (41), and (42), the span loading for the wing-body combination is plotted for several ratios of gap to semispan in figure 1(a). No lift is induced on the body aft of the trailing edge of the wing, according to slender-body theory.

## Pressure Distribution on Body

The expression for the loading coefficient $\Delta p/q$ on the body is obtained in the same manner as for the wing in a preceding section. The result is

$$\left(\frac{\Delta p}{q\alpha}\right)_B = 4m\left[1+\left(\frac{r_o}{s}\right)^2\right]\left[1-\frac{E(k)}{K(k)}\right]\left[\frac{\left(1-\frac{r_o^2}{s^2}\right)\sin\psi_2}{\sqrt{\left(1+\frac{r_o^2}{s^2}\right)^2-4\left(\frac{y}{s}\right)^2}}-\frac{Z(\psi_2,k)}{k^2}\right] \tag{43}$$

If $t = r_o$, equation (43) reduces to the result given in reference 8 for the lifting pressure coefficient on an infinite cylindrical body in the presence of a lifting low-aspect-ratio wing.

The distribution of lifting pressures on a spanwise strip for the present wing-body combination is shown for several ratios of gap to semispan in figure 2(a). For comparison, the pressure distribution of the wing alone is also shown in the same figure.

## Lift of Body

The lift induced on the body by the wing can be found by the formula

$$L_B = \int_{-r_o}^{r_o} \frac{dL_B}{dy}\,dy = \int_o^{r_1} \frac{dL_B}{dy_1}\,dy_1 \tag{44}$$

with $dL_B/dy_1$ given in equation (41) above. The final result can be written

$$\left(\frac{L}{q\alpha}\right)_B = \pi\left\{t_1^2+(s_{o1}^2-r_1^2)\left[1-2\frac{E(k)}{K(k)}\right]\right\}\left[1-\Lambda_o(\psi,k_1)\right]-\pi r_1^2-$$

$$2t_1\sqrt{s_{o1}^2-r_1^2}\left\{E(k_1)-K(k_1)+\frac{r_1^2}{t_1^2}\left[1-2\frac{E(k)}{K(k)}\right]K(k_1)\right\} \tag{45}$$

where $\psi$, $k$, and $k_1$ are evaluated for $s_1 = s_{o1}$.

For the lift of the wing-body combination (exclusive of the nose), equation (45) is added to equation (35). This yields the simple equation

$$\left(\frac{L}{q\alpha}\right)_{WB} = 2\pi\left\{(t_1^2-r_1^2) + (s_{o1}^2-r_1^2)\left[1 - 2\frac{E(k)}{K(k)}\right]\right\} \qquad (46)$$

When written in terms of the physical dimensions, equation (46) agrees with equation (2) of reference 4.

The total lift of the body and of the wing-body combination can be obtained by simply adding the lift of the nose (which has the value $\frac{\pi}{2} r_1^2 = 2\pi r_o^2$) to the respective expressions for lift.

For the case in which the gap is zero ($t_1 = r_1$), equation (46) reduces to the result given for this case in reference 8.

Equations (35), (45), and (46) have been evaluated for several gap semispan ratios; the results, given as ratios of lift in the presence of a gap to the lift when the gap is zero, are shown in figures 3(a) and 4(a).

## PROBLEM 2 - BODY AT ZERO ANGLE OF ATTACK TO FREE STREAM, WING DEFLECTED AT SMALL ANGLE WITH RESPECT TO BODY

The second problem is complicated by the fact that the total gap between the wing panel and the cylindrical body is the sum of two gaps, one of which is the constant clearance required for mechanical reasons, and the other is the gap caused by wing deflection. The latter gap distance varies from apex to hinge line, and from hinge line to trailing edge of each wing panel, and also changes in magnitude with the angle of deflection. However, for the small deflections considered here (say of the order of $5^\circ$), the gap created by deflecting the wing will be small for the usual ratios of body diameter to wing chord encountered in practice, and the total gap distance will be taken as a distance independent of wing deflection angle.

### Boundary Conditions

Certain boundary conditions for this problem differ from those of the previous problem, although the basic conditions, such as the zero

value for the jump in potential at the leading edge of the wing panel, apply here as well. Stated first in the physical yz plane (see sketch (b)), the salient boundary conditions for this problem are

$$\left.\begin{array}{ll} \text{(i)}\ v_r = 0 & \rho = r_o,\ 0 \le \theta \le 2\pi \\ \text{(ii)}\ \Delta v = 0 & z = 0,\ r_o^2 < y^2 < t^2 \\ \text{(iii)}\ w = -V_o\delta & z = 0,\ t^2 < y^2 < s^2 \\ \text{(iv)}\ v = 0,\ w = 0 & \rho = \infty,\ 0 \le \theta \le 2\pi \\ \text{(v)}\ \Delta\varphi = 0 & y^2 \ge s^2 \end{array}\right\} \quad (47)$$

Transforming equations (47) to the $\xi_1$ plane by means of equations (6) gives

$$\left.\begin{array}{ll} \text{(i)}\ w_1 = 0 & z_1 = 0,\ 0 < y_1^2 < r_1^2 \\ \text{(ii)}\ \Delta v_1 = 0 & z_1 = 0,\ r_1^2 < y_1^2 < t_1^2 \\ \text{(iii)}\ w_1 = -\dfrac{V_o\delta}{2}\left(1 + \dfrac{y_1}{\sqrt{y_1^2 - r_1^2}}\right) & t_1^2 < y_1^2 < s_1^2 \\ \text{(iv)}\ v_1 = 0,\ w_1 = 0 & \rho_1 = \infty,\ 0 \le \theta \le 2\pi \\ \text{(v)}\ \Delta\varphi_1 = 0 & y_1^2 \ge s_1^2 \end{array}\right\} \quad (48)$$

## Solution of the Integral Equation

Commencing with equation (13) we shall obtain a solution for $\Delta v_1$, the jump in lateral velocity along the real axis. In the region $0 < \eta_1 < r_1^2$ the vertical component of velocity is zero so that equation (13) can be written

$$\frac{1}{2\pi}\int_{t_1^2}^{s_1^2} \frac{\Delta v_{1W}}{\eta_1 - \eta_2}\, d\eta_2 = -\frac{1}{2\pi}\int_{0}^{r_1^2} \frac{\Delta v_{1B}}{\eta_1 - \eta_2}\, d\eta_2;\quad 0 < \eta_1 < r_1^2 \quad (49)$$

Applying the inversion formulas given by equations (15) and (17) gives

$$\Delta v_{1B}(\eta_2) = \frac{1}{\pi^2}\sqrt{\frac{\eta_2}{r_1^2 - \eta_2}}\int_{0}^{r_1^2} \frac{d\eta_1}{\eta_2 - \eta_1}\sqrt{\frac{r_1^2 - \eta_1}{\eta_1}}\int_{t_1^2}^{s_1^2} \frac{\Delta v_{1W}(\eta_3)}{\eta_1 - \eta_3}\, d\eta_3 \quad (50)$$

or after integration with respect to $\eta_1$

$$\Delta v_{1B}(\eta_2) = \frac{1}{\pi}\sqrt{\frac{\eta_2}{r_1^2-\eta_2}}\int_{t_1^2}^{s_1^2}\frac{\Delta v_{1W}(\eta_3)}{\eta_2-\eta_3}\sqrt{\frac{\eta_3-r_1^2}{\eta_3}}\,d\eta_3 \tag{51}$$

Next, substitution of equation (51) into equation (13) will eliminate $\Delta v_{1B}$ and provide an integral equation for the vorticity across the wing. Thus

$$w_1(\eta_1) = -\frac{1}{2\pi^2}\int_0^{r_1^2}\frac{d\eta_2}{\eta_1-\eta_2}\sqrt{\frac{\eta_2}{r_1^2-\eta_2}}\int_{t_1^2}^{s_1^2}\frac{\Delta v_{1W}(\eta_3)}{\eta_2-\eta_3}\sqrt{\frac{\eta_3-r_1^2}{\eta_3}}\,d\eta_3 - \frac{1}{2\pi}\int_{t_1^2}^{s_1^2}\frac{\Delta v_{1W}}{\eta_1-\eta_2}\,d\eta_2 \tag{52}$$

Integrating the double integral first with respect to $\eta_2$ we find that for $t_1^2 \le \eta_1 \le s_1^2$, equation (52) reduces to

$$w_1(\eta_1)\sqrt{\frac{\eta_1-r_1^2}{\eta_1}} = -\frac{1}{2\pi}\int_{t_1^2}^{s_1^2}\frac{\Delta v_{1W}(\eta_3)}{\eta_1-\eta_3}\sqrt{\frac{\eta_3-r_1^2}{\eta_3}}\,d\eta_3;\quad t_1^2 \le \eta_1 \le s_1^2 \tag{53}$$

Equation (53) can be inverted by the formulas given by equations (15) and (16). Thus is obtained

$$\Delta v_{1W} = \frac{2\sqrt{\eta_1}}{\pi\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}}\left[C + \int_{t_1^2}^{s_1^2}\frac{w_1(\eta_2)}{\eta_1-\eta_2}\sqrt{\frac{(s_1^2-\eta_2)(\eta_2-t_1^2)(\eta_2-r_1^2)}{\eta_2}}\,d\eta_2\right] \tag{54}$$

where

$$C = \frac{1}{2}\int_{t_1^2}^{s_1^2} \Delta v_{1W}\sqrt{\frac{\eta_3 - r_1^2}{\eta_3}}\,d\eta_3$$

As in problem 1, the value of the constant $C$ can be determined from the condition that $\Delta\varphi_1(t_1) = 0$. Since

$$\Delta\varphi_1(t_1) = \int_{s_1^2}^{t_1^2} \frac{\Delta v_{1W}}{2\sqrt{\eta_1}}\,d\eta_1$$

and from boundary conditions

$$w_1(\eta_2) = -\frac{V_0\delta}{2}\left(1 + \sqrt{\frac{\eta_2}{\eta_2 - r_1^2}}\right)$$

it follows from equation (54) that

$$C\int_{s_1^2}^{t_1^2} \frac{d\eta_1}{\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}} = \frac{V_0\delta}{2}\int_{s_1^2}^{t_1^2} \frac{d\eta_1}{\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}}$$

$$\left[\int_{s_1^2}^{t_1^2} \frac{d\eta_2}{\eta_1-\eta_2}\sqrt{\frac{(s_1^2-\eta_2)(\eta_2-t_1^2)(\eta_2-r_1^2)}{\eta_2}} + \int_{s_1^2}^{t_1^2} \frac{d\eta_2}{\eta_1-\eta_2}\sqrt{(s_1^2-\eta_2)(\eta_2-t_1^2)}\right] \tag{55}$$

from which

$$C = \frac{V_0\delta\sqrt{s_1^2-r_1^2}}{4K(k)} \int_{s_1^2}^{t_1^2} \frac{d\eta_1}{\sqrt{(s_1^2-\eta_1)(\eta_1-t_1^2)(\eta_1-r_1^2)}}$$

$$\left[ \int_{s_1^2}^{t_1^2} \frac{d\eta_2}{\eta_1 - \eta_2} \sqrt{\frac{(s_1^2-\eta_2)(\eta_2-t_1^2)(\eta_2-r_1^2)}{\eta_2}} - \frac{\pi}{2}(2\eta_1 - s_1^2 - t_1^2) \right] \tag{56}$$

This expression is left in this form since further integration is not advantageous at this time.

## Span Loading on Wing Panel

The circulation $\Delta\varphi_1(y_1)$ will be obtained by integrating the expression for $\Delta v_{1W}$ given by equation (54); that is

$$\Delta\varphi_{1W}(y_1) = \int_{s_1^2}^{y_1^2} \frac{\Delta v_{1W}}{2\sqrt{\eta_1}} d\eta_1; \qquad \eta_1 = y_1^2 \tag{57}$$

By a suitable choice of the order of integration, equation (57) can be evaluated to give

$$\Delta\varphi_{1W}(y_1) = V_0\delta \left\{ \frac{2}{\pi} K(k_1) \left[ \frac{r_1^2}{t_1} Z(\psi_1,k) - y_1 Z(\psi_3,k_1) \right] + \right.$$

$$\left. \sqrt{s_1^2-r_1^2}\, Z(\psi_1,k) \left[ 1 + \Lambda_0(\psi,k_1) \right] \right\} \tag{58}$$

where

$$\psi_3 = \sin^{-1}\sqrt{\frac{(s_1^2-y_1^2)t_1^2}{(s_1^2-t_1^2)y_1^2}}$$

In the case of zero gap ($t_1 = r_1$), equation (58) degenerates into

$$\Delta\varphi_{1W(t_1=r_1)} = \frac{V_0\delta}{\pi}\left\{\pi\sqrt{s_1^2-y_1^2}\left[1+\frac{2}{\pi}\cos^{-1}\frac{r_1}{s_1}\right]+ y_1\ln\frac{\sqrt{s_1^2-r_1^2}+\frac{r_1}{y_1}\sqrt{s_1^2-y_1^2}}{\sqrt{s_1^2-r_1^2}-\frac{r_1}{y_1}\sqrt{s_1^2-y_1^2}} - r_1\ln\frac{\sqrt{s_1^2-r_1^2}+\sqrt{s_1^2-y_1^2}}{\sqrt{s_1^2-r_1^2}-\sqrt{s_1^2-y_1^2}}\right\} \tag{59}$$

Equation (59) agrees with an unpublished result obtained by Gaynor J. Adams, who solved a similar problem of a slender wing-body combination having the wing deflected - no gap being considered.

Since $\Delta\varphi = 0$ at the leading edge, the expression for the span loading can be written

$$\frac{\frac{dL}{dy}}{4\alpha q s_0^*} = \frac{1}{\pi}\left(1+\frac{g}{s_0^*}\right)\left(\left\{4\frac{t}{s_0}\frac{\frac{r_0^2}{s_0^2}}{\frac{t^2}{s_0^2}+\frac{r_0^2}{s_0^2}}K(k_1)+\frac{\pi}{2}\left(1-\frac{r_0^2}{s_0^2}\right)\left[1+\Lambda_0(\psi,k)\right]\right\}Z(\psi_1,k) - \frac{s_0}{y}\left(\frac{y^2}{s_0^2}+\frac{r_0^2}{s_0^2}\right)K(k_1)\,Z(\psi_3,k_1)\right) \tag{60}$$

where the moduli and arguments of the elliptic integrals are evaluated for $s_1 = s_{01}$.

## Pressure Distribution on Wing Panels

Following the procedure outlined in problem 1, page 11, we solve for the loading coefficient of the wing panels. Differentiation of equation (58) with respect to $s_1$ yields

$$\frac{\partial \Delta\varphi_{1W}}{\partial s_1} = \frac{V_o \delta}{\pi} \frac{s_1}{s_1^2 - t_1^2} \left( \left[ 1 - \frac{E(k)}{K(k)} \right] \left\{ \pi \sqrt{s_1^2 - r_1^2} \left[ 1 + \Lambda_o(\psi, k_1) \right] + \frac{2r_1^2}{t_1} K(k_1) \right\} - 2t_1 \left[ K(k_1) - E(k_1) \right] \right) \left[ Z(\psi_1, k) + \sqrt{\frac{(y_1^2 - t_1^2)(y_1^2 - r_1^2)}{(s_1^2 - y_1^2)(s_1^2 - r_1^2)}} \right] \tag{61}$$

and the expression for the loading coefficient becomes

$$\left( \frac{\Delta p}{q\delta} \right)_W = \frac{2m}{\pi} \frac{(s^4 - r_o^4)t^2}{s(s^2 - t^2)(s^2 t^2 - r_o^4)} \left[ \frac{s^2(y^2 - r_o^2)}{yt(s^2 - r_o^2)} \sqrt{\frac{(y^2 t^2 - r_o^4)(y^2 - t^2)}{(s^2 y^2 - r_o^4)(s^2 - y^2)}} + Z(\psi_1, k) \right] \left( \left[ 1 - \frac{E(k)}{K(k)} \right] \left\{ \frac{8 r_o^2 t}{t^2 + r_o^2} K(k_1) + \frac{\pi(s^2 - r_o^2)}{s} \left[ 1 + \Lambda_o(\psi, k_1) \right] \right\} - \frac{2(t^2 + r_o^2)}{t} \left[ K(k_1) - E(k_1) \right] \right) \tag{62}$$

For the limiting case of zero gap, equation (62) degenerates to

$$\left(\frac{\Delta p}{q\delta}\right)_{W(t=r_o)} = \frac{8m}{\pi} \frac{\left[1 - \left(\frac{r_o}{s}\right)^4\right] \tan^{-1}\left(\frac{s}{r_o}\right)}{\sqrt{\left[1 + \left(\frac{r_o}{s}\right)^2\right]^2 - \left[\frac{y}{s} + \frac{s}{y}\left(\frac{r_o}{s}\right)^2\right]^2}} \tag{63}$$

## Lift of Wing Panels

The solution for the lift of the wing panels in this problem follows the same pattern as adopted in the first problem. In the $\xi_1$ plane

$$\left(\frac{L}{q}\right)_W = \frac{2}{V_o} \int_{t_1}^{s_{o1}} (\Delta\varphi)_{TE} \left(1 + \frac{y_1}{\sqrt{y_1^2 - r_1^2}}\right) dy_1 \tag{64}$$

Substitute the expression for $\Delta\varphi_{TE}$ obtained from equation (58) into equation (64) and integrate and there results

$$\left(\frac{L}{q\delta}\right)_W = \frac{\pi}{2}\left\{(s_{o1}^2 - r_1^2)\left[1 - 2\frac{E(k)}{K(k)}\right] + t_1^2\right\}\left[1 + \Lambda_o(\psi, k_1) + \frac{2r_1^2 K(k_1)}{\pi t_1 \sqrt{s_{o1}^2 - r_1^2}}\right]^2 +$$

$$2t_1\sqrt{s_{o1}^2 - r_1^2}\left[E(k_1) - \frac{K(k_1)}{\sin^2\psi}\right]\left[1 + \Lambda_o(\psi, k_1) + \frac{r_1^2 K(k_1)}{\pi t_1 \sqrt{s_{o1}^2 - r_1^2}}\right] +$$

$$2\frac{r_1^2}{\pi} E(k_1)K(k_1) - \frac{\pi}{2} r_1^2 \tag{65}$$

where the moduli and arguments of the elliptic integrals are evaluated using $s_1 = s_{o1}$.

Span Loading on the Body

Equation (51) provides an expression for the jump in lateral velocity $\Delta v_{1B}$ along the real axis in the transformed plane between 0 and $r_1^2$ in terms of $\Delta v_{1W}$ between $t_1^2$ and $s_1^2$. Combining equations (51) and (54) gives the expression

$$\Delta v_{1B}(\eta_1) = \frac{2}{\pi^2}\sqrt{\frac{\eta_1}{r_1^2-\eta_1}}\int_{t_1^2}^{s_1^2}\frac{d\eta_2}{(\eta_1-\eta_2)\sqrt{(s_1^2-\eta_2)(\eta_2-t_1^2)}}\left[C + \int_{t_1^2}^{s_1^2}\frac{w_1(\eta_3)}{(\eta_2-\eta_3)}\sqrt{\frac{(s_1^2-\eta_3)(\eta_3-t_1^2)(\eta_3-r_1^2)}{\eta_3}}\,d\eta_3\right] \tag{66}$$

An expression for $\Delta\Phi_{1B}$ can now be found by substituting equation (66) into

$$\Delta\Phi_{1B} = \int_{r_1^2}^{y_1^2}\frac{\Delta v_{1B}}{2\sqrt{\eta_1}}\,d\eta_1$$

Using the expression for C given in equation (56) and performing the integrations gives

$$\Delta\Phi_{1B} = \frac{V_0\delta}{\pi}\left\{2K(k_1)\left[\frac{1}{t_1}\sqrt{\frac{r_1^2-y_1^2}{s_1^2-r_1^2}}\left(r_1^2\sqrt{\frac{s_1^2-y_1^2}{t_1^2-y_1^2}} - s_1^2\sqrt{\frac{t_1^2-y_1^2}{s_1^2-y_1^2}}\right) - \frac{r_1^2}{t_1}Z(\psi_2,k) - y_1 Z(\psi_4,k_1)\right] - \pi\sqrt{r_1^2-y_1^2} + \pi\left[1+\Lambda_0(\psi,k_1)\right]\left[\sqrt{s_1^2-y_1^2}\sin\psi_2 - \sqrt{s_1^2-r_1^2}\,Z(\psi_2,k)\right]\right\} \tag{67}$$

where

$$\psi_4 = \sin^{-1}\sqrt{\frac{(s_1^2-r_1^2)y_1^2}{(s_1^2-y_1^2)r_1^2}}$$

With $s_1 = s_{o1}$, equation (67) gives the value of $\Delta\Phi_B$ at the trailing edge of the wing. If the body extends aft of the trailing edge of the wing, the present boundary conditions cannot properly be applied to find the loading on the afterbody. No attempt to determine such loading will be made here.

Since $(\Delta\Phi_{1B})_{LE}$ is zero,

$$\frac{dL_B}{dy_1} = \frac{2q\delta}{\pi}\left\{2K(k_1)\left[\frac{1}{t_1}\sqrt{\frac{r_1^2-y_1^2}{s_{o1}^2-r_1^2}}\left(r_1^2\sqrt{\frac{s_1^2-y_1^2}{t_1^2-y_1^2}} - s_1^2\sqrt{\frac{t_1^2-y_1^2}{s_1^2-y_1^2}}\right) - \frac{r_1^2}{t_1}Z(\psi_2,k) - y_1 Z(\psi_4,k_1)\right] - \pi\sqrt{r_1^2-y_1^2} + \pi\left[1+\Lambda_0(\psi,k_1)\right]\left[\sqrt{s_{o1}^2-y_1^2}\sin\psi_2 - \sqrt{s_{o1}^2-r_1^2}\,Z(\psi_2,k)\right]\right\} \tag{68}$$

where $s_{o1}$, the value of $s_1$ at the trailing edge, is used throughout. For the condition of zero gap, the expression for span loading can be shown to be

$$\left(\frac{dL_B}{dy_1}\right)_{t_1=r_1} = \frac{2q\delta}{\pi}\left[\pi\left(\sqrt{s_{o1}^2-y_1^2} - \sqrt{r_1^2-y_1^2}\right) + 2\sqrt{s_{o1}^2-y_1^2}\cos^{-1}\left(\frac{r_1}{s_{o1}}\right) + y_1 \ln\left(\frac{r_1\sqrt{s_{o1}^2-y_1^2} + y_1\sqrt{s_{o1}^2-r_1^2}}{r_1\sqrt{s_{o1}^2-y_1^2} - y_1\sqrt{s_{o1}^2-r_1^2}}\right) - r_1 \ln\left(\frac{\sqrt{s_{o1}^2-y_1^2} + \sqrt{s_{o1}^2-r_1^2}}{\sqrt{s_{o1}^2-y_1^2} - \sqrt{s_{o1}^2-r_1^2}}\right)\right] \tag{69}$$

When equations (68) and (60) are evaluated for several ratios of gap to semispan, the results shown in figure 1(b) are obtained for the span loading on the wing-body combination.

## Pressure Distribution on Body

The differentiation of $\Delta\varphi_{1B}$ (eq. (67)) yields

$$\frac{\partial\Delta\varphi_{1B}}{\partial s_1} = \frac{2V_o\delta}{\pi}\frac{s_1 t_1}{(s_1^2-t_1^2)}\left(K(k_1) - E(k_1) - \left[1 - \frac{E(k)}{K(k)}\right]\left\{\frac{r_1^2}{t_1^2}K(k_1) + \frac{\pi}{2t_1}\sqrt{s_1^2-r_1^2}\left[1 + \Lambda_o(\psi,k_1)\right]\right\}\right)\left[Z(\psi_2,k) - \frac{k\sin\psi_2}{\sin\psi_1}\right] \tag{70}$$

so that

$$\left(\frac{\Delta p}{q\delta}\right)_B = \frac{4m}{\pi}\frac{t(s^4-r_o^4)(t^2+r_o^2)}{s(s^2-t^2)(s^2t^2-r_o^4)}\left(K(k_1) - E(k_1) - \left[1 - \frac{E(k)}{K(k)}\right]\left\{\frac{4r_o^2t^2K(k_1)}{(t^2+r_o^2)^2} + \frac{\pi}{2}\frac{t(s^2-r_o^2)}{s(t^2+r_o^2)}\left[1 + \Lambda_o(\psi,k_1)\right]\right\}\right)\left[Z(\psi_2,k) - \frac{k\sin\psi_2}{\sin\psi_1}\right] \tag{71}$$

If $t$ is set equal to $r_o$ in equation (71), the loading coefficient for the case of zero gap is obtained as

$$\left(\frac{\Delta p}{q\delta}\right)_B = \frac{8m}{\pi}\frac{\left[1 - \left(\frac{r_o}{s}\right)^4\right]\tan^{-1}\left(\frac{s}{r_o}\right)}{\sqrt{\left[1 + \left(\frac{r_o}{s}\right)^2\right]^2 - 4\left(\frac{y}{s}\right)^2}} \tag{72}$$

The loadings on a spanwise strip of the wing-body combination, obtained from equations (71) and (62), are shown graphically in figure 2(b) for several ratios of gap to semispan.

## Lift of Body

The lift of the body is found in the same manner as in the first problem, namely, by integrating the span loading over the diameter. The integration is carried out in the $\xi_1$ plane for convenience. When equation (68) is substituted into the equation

$$L_B = 2\int_0^{r_1} \frac{dL_B}{dy_1}\, dy_1$$

the expression for lift on the body is

$$\left(\frac{L}{q\delta}\right)_B = \frac{\pi}{2}\left\{(s_{o1}^2-r_1^2)\left[1-2\frac{E(k)}{K(k)}\right]+t_1^2\right\}\left\{1-\left[\Lambda_0(\psi,k_1)+\frac{2r_1^2K(k_1)}{\pi t_1\sqrt{s_{o1}^2-r_1^2}}\right]^2\right\} - 2t_1\sqrt{s_{o1}^2-r_1^2}\left[E(k_1)-\frac{K(k_1)}{\sin^2\psi}\right]$$

$$\left[\Lambda_0(\psi,k_1)+\frac{r_1^2K(k_1)}{\pi t_1\sqrt{s_{o1}^2-r_1^2}}\right]-\frac{2r_1^2}{\pi}E(k_1)K(k_1)-\frac{\pi}{2}r_1^2 \qquad (73)$$

with the moduli and arguments of the elliptic integrals evaluated for $s_1 = s_{o1}$.

The expression for the lift of the wing-body combination in the presence of a gap is obtained by adding equations (73) and (65) and is given by

$$\left(\frac{L}{q\delta}\right)_{WB} = \pi\left\{t_1^2 + (s_{o1}^2 - r_1^2)\left[1 - 2\frac{E(k)}{K(k)}\right]\right\}\left\{1 + \Lambda_o(\psi,k_1)\right\} - \pi r_1^2 +$$

$$2t_1\sqrt{s_{o1}^2 - r_1^2}\left\{E(k_1) - K(k_1) + \frac{r_1^2}{t_1^2}\left[1 - 2\frac{E(k)}{K(k)}\right]K(k_1)\right\} \quad (74)$$

where $s_1 = s_{o1}$ is used throughout.

As previously mentioned, no effect of the afterbody is included in the analysis of the lifting properties of the body, so equations (73) and (74) are strictly applicable only to wing-body combinations having no afterbody.

It is interesting to note that the lift of the wing-body combination per unit angle of wing deflection is identical to the lift of the wing of the combination (exclusive of body nose) per unit angle of attack (cf. eqs. (74) and (35)).

For the case of zero gap ($t_1 = r_1$), equation (74) may be written

$$C_{L_{WB}} = \frac{1}{2} C_{L_W}\left[1 - \left(\frac{r_o}{s_o}\right)^2\right]^2\left[1 - \frac{4\frac{r_o}{s_o}}{\pi\left(1 - \frac{r_o^2}{s_o^2}\right)} + \right.$$

$$\left. \frac{2}{\pi}\left(\frac{1 + \frac{r_o^2}{s_o^2}}{1 - \frac{r_o^2}{s_o^2}}\right)^2 \cos^{-1}\frac{2\frac{r_o}{s_o}}{1 + \frac{r_o^2}{s_o^2}}\right] \quad (75)$$

where $C_{L_W} = \frac{\pi}{2} A\delta$ = lift coefficient of a slender triangular wing. Equation (75) agrees with a result obtained in reference 10.

Figures 3(b) and 4(b) present the variation of lift with gap width for the wing, body, and wing-body combination for ratios of body radius to wing semispan equal to 0.5 and 0.216.

## EFFECTIVENESS OF ALL-MOVABLE WING

The effectiveness of a control surface in producing lift is often of interest to designers of aircraft. It is usually expressed in terms of an effectiveness parameter defined by

$$-\frac{\alpha}{\delta} = \left( \frac{\frac{dC_L}{d\delta}}{\frac{dC_L}{d\alpha}} \right)_{C_L=\text{constant}}$$

For the wing-body combination investigated in this report, the variation of the effectiveness parameter with gap width is shown in figure 5. Figure 5(a) shows results obtained by omitting the lift of the nose, which according to theory is invariant with gap, and by omitting the lift of any afterbody, which cannot be calculated in the present analysis. In figure 5(b) the nose lift has been included in the calculations, so that the latter figure pertains to a slender wing-body combination having no afterbody.

## DISCUSSION OF RESULTS

Application of slender-body theory to the determination of the lift of a slender wing-body combination having a gap between wing and body indicates that comparatively large losses in lifting pressures, circulation, and lift are caused by even minute gaps; however, it is anticipated that these large losses will not be realized in practice for very small gaps because of the viscous and possibly the compressible properties of actual fluids. Experimental investigation is necessary to determine the range of gap dimensions and of angles of attack for which the simplified theory is valid.

If attention is confined to the wing-body combination exclusive of nose and afterbody, it is evident from figures 3 and 4 that for a given gap width the percent loss in lift is greater when both wing and body are at an angle of attack than when only the wing is set at an angle of incidence. Comparison of figure 3 with figure 4 shows that the percent loss in lift due to gap increases with increasing ratios of body radius to wing semispan. Figure 5(a) indicates that the presence of a gap tends to reduce the difference between the lift of the wing-body combination due to angle of attack and that caused by wing deflection. The influence of the ratio of body radius to wing semispan is such that for larger ratios the effectiveness of the wing as a control surface is diminished.

The effect of the lift of the nose in the angle-of-attack case is to reduce the percent loss in lift of the complete configuration for a given gap width from that given for the portion of the configuration above; thus the effectiveness of the wing as a control surface is lowered from the values given in figure 5(a) to those shown in figure 5(b). It appears from figure 5(b) that, with no afterbody, the control effectiveness of the deflected wing may increase or decrease with increasing gap width accordingly as the ratio of body radius to wing semispan is small or large.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., May 12, 1954

## APPENDIX A

## NOTATION

$A$ aspect ratio, $4\frac{s_o^2}{S}$

$C_L$ lift coefficient, $\frac{L}{qS}$

$E(k)$ complete elliptic integral of the second kind with modulus $k$, $\int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\, d\theta$

$E(\psi,k)$ incomplete elliptic integral of the second kind with argument $\psi$ and modulus $k$, $\int_0^{\psi} \sqrt{1 - k^2\sin^2\theta}\, d\theta$

$F(\psi,k)$ incomplete elliptic integral of the first kind with argument $\psi$ and modulus $k$, $\int_0^{\psi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$

$g$ gap width between fuselage and wing panels, $t - r_o$

$k$, $k_1$ moduli of elliptic integrals, $k = \sqrt{\frac{s_1^2 - t_1^2}{s_1^2 - r_1^2}}$; $k_1 = \frac{r_1}{t_1} k$

$k'$ complementary modulus, $\sqrt{1 - k^2}$

$K(k)$ complete elliptic integral of the first kind with modulus $k$, $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$

$L$ lift force

$m$ slope of leading edge of wing panel, triangular wings, $\frac{ds}{dx}$

$M_o$ free-stream Mach number

| | |
|---|---|
| $\frac{\Delta p}{q}$ | loading coefficient, $\frac{p_l - p_u}{q}$ |
| $q$ | free-stream dynamic pressure, $\frac{1}{2}\rho_o V_o^2$ |
| $r_o$ | radius of fuselage (cylindrical portion) |
| $s$ | local semispan of wing |
| $s_o$ | maximum semispan of wing |
| $s^*$, $s_o^*$ | local and maximum semispan of wing, respectively, when gap is zero |
| $S$ | combined areas of both wing panels |
| $t$ | distance from body axis to inside edge of wing panel |
| $u,v,w$ | perturbation velocity components in the $x,y,z$ directions, respectively |
| $\Delta u$, $\Delta v$ | jump in velocities across the $z = 0$ plane $(u_u - u_l, v_u - v_l)$ |
| $v_r$ | radial component of perturbation velocity in a plane perpendicular to the $x$ axis |
| $V_o$ | free-stream velocity |
| $x,y,z$ | Cartesian coordinates |
| $\alpha$ | angle of attack of body axis |
| $\Gamma$ | total circulation about a wing section |
| $\delta$ | wing deflection angle with respect to body axis |
| $\epsilon$ | apex angle of wing panel |
| $Z(\psi,k)$ | Jacobi's zeta function, $E(\psi,k) - \frac{E(k)}{K(k)} F(\psi,k)$ |
| $\eta$ | $y^2$ |
| $\theta$ | polar coordinate in a plane perpendicular to the $x$ axis |
| $\Lambda_o(\psi,k)$ | Heuman's lambda function, $\frac{2}{\pi}\left\{[E(k) - K(k)]\, F(\psi,k') + K(k)E(\psi,k')\right\}$ |
| $\xi$ | complex variable, $y + iz$ |

$\rho$ polar distance in a plane perpendicular to the x axis

$\rho_0$ free-stream density

$\varphi$ perturbation velocity potential

$\Delta\varphi$ jump in $\varphi$ across the $z = 0$ plane

$\psi, \psi_1, \psi_2, \psi_3, \psi_4$ arguments of elliptic integrals, $\psi = \sin^{-1}\sqrt{\frac{s_1^2 - r_1^2}{s_1^2}}$;

$$\psi_1 = \sin^{-1}\sqrt{\frac{s_1^2 - y_1^2}{s_1^2 - t_1^2}};\quad \psi_2 = \sin^{-1}\sqrt{\frac{r_1^2 - y_1^2}{t_1^2 - y_1^2}};$$

$$\psi_3 = \sin^{-1}\sqrt{\frac{(s_1^2 - y_1^2)t_1^2}{(s_1^2 - t_1^2)y_1^2}};\quad \psi_4 = \sin^{-1}\sqrt{\frac{(s_1^2 - r_1^2)y_1^2}{(s_1^2 - y_1^2)r_1^2}}$$

## Subscripts

1 complex plane resulting from application of Joukowski transformation to the physical complex plane (except when used in the elliptic integrals)

B body

l lower surface of $z = 0$ plane or of wing panel

u upper surface of $z = 0$ plane or of wing panel

W wing

WB wing-body combination

## REFERENCES


1. von Kármán, Th., and Burgers, J. M.: General Aerodynamic Theory - Perfect Fluids. Vol. II of Aerodynamic Theory, div. E, W. F. Durand, ed., Julius Springer (Berlin), 1934.

2. Lomax, Harvard, and Byrd, Paul F.: Theoretical Aerodynamic Characteristics of a Family of Slender Wing-Tail Body Combinations. NACA TN 2554, 1951.

3. Bleviss, Z. O., and Struble, R. A.: Some Aerodynamic Effects of Streamwise Gaps in Low-Aspect-Ratio Lifting Surfaces at Supersonic Speeds. IAS Preprint No. 396, 1953.

4. Mirels, H.: Gap Effect on Slender Wing-Body Interference. Readers' Forum, Jour. Aero. Sci., vol. 20, no. 8, Aug. 1953.

5. Munk, Max M., and Cario, Gunther.: Flugel mit Spalt in Fahrtrichtung. Technische Berichte, Bd. 1, 1917, Charlottenburg, p. 219.

6. Byrd, Paul F., and Friedman, Morris D.: Handbook of Elliptic Integrals for Engineers and Physicists. Berlin, Springer-Verlag, 1954.

7. Milne-Thompson, L. M.: Jacobian Elliptic Function Tables. Dover Pub., N. Y., 1950.

8. Spreiter, John R.: Aerodynamic Forces on Slender Plane- and Cruciform-Wing and Body Combinations. NACA Rep. 962, 1950. (Formerly NACA TN's 1662 and 1897)

9. Heuman, Carl: Tables of Complete Elliptic Integrals. Jour. Math. and Phys., vol. 19-20, 1940-41, pp. 127-206.

10. Heaslet, Max. A., and Spreiter, John R.: Reciprocity Relations in Aerodynamics. NACA Rep. 1119, 1953. (Formerly NACA TN 2700)

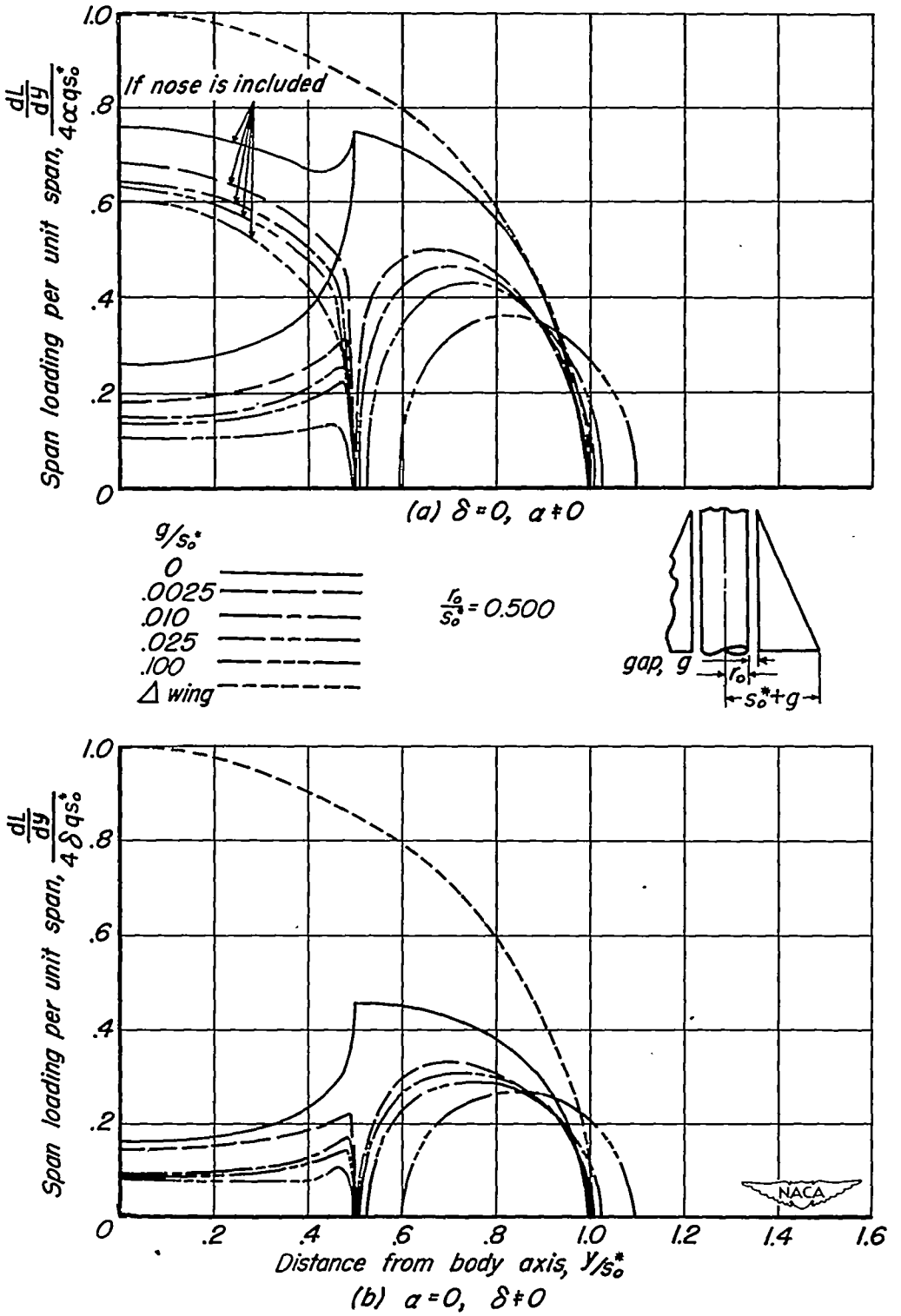
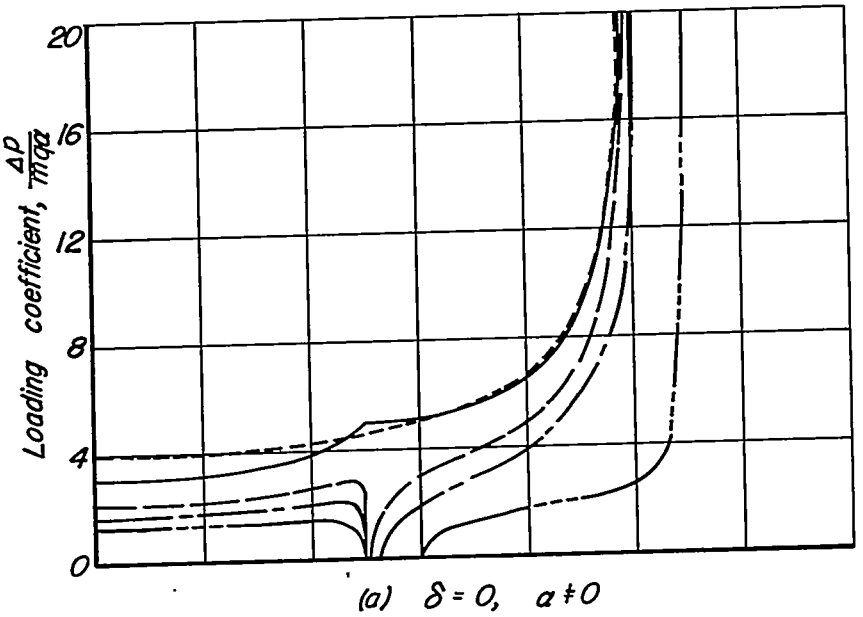
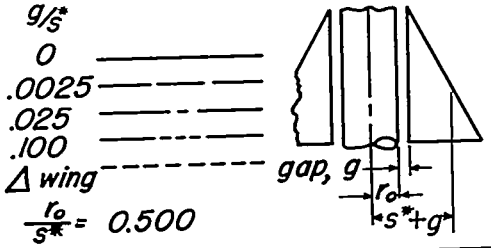
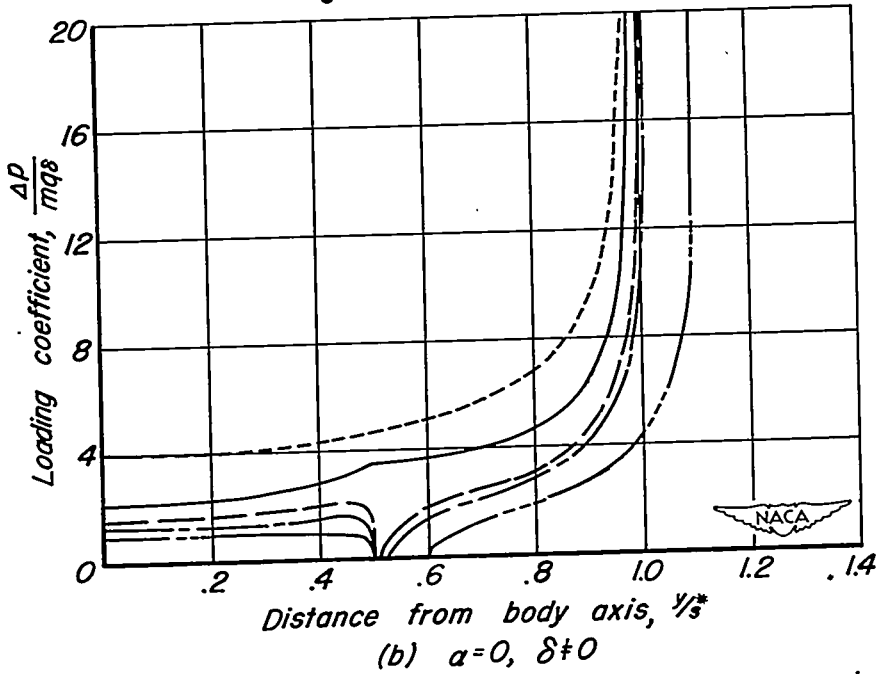



*Figure 1.– Span loading on a wing-body combination for several values of gap-semispan ratio.*




(a) $\delta = 0$, $\alpha \neq 0$




(b) $\alpha = 0$, $\delta \neq 0$

Figure 2.- Lifting pressure distribution on a spanwise strip of a wing-body combination for several values of gap width.

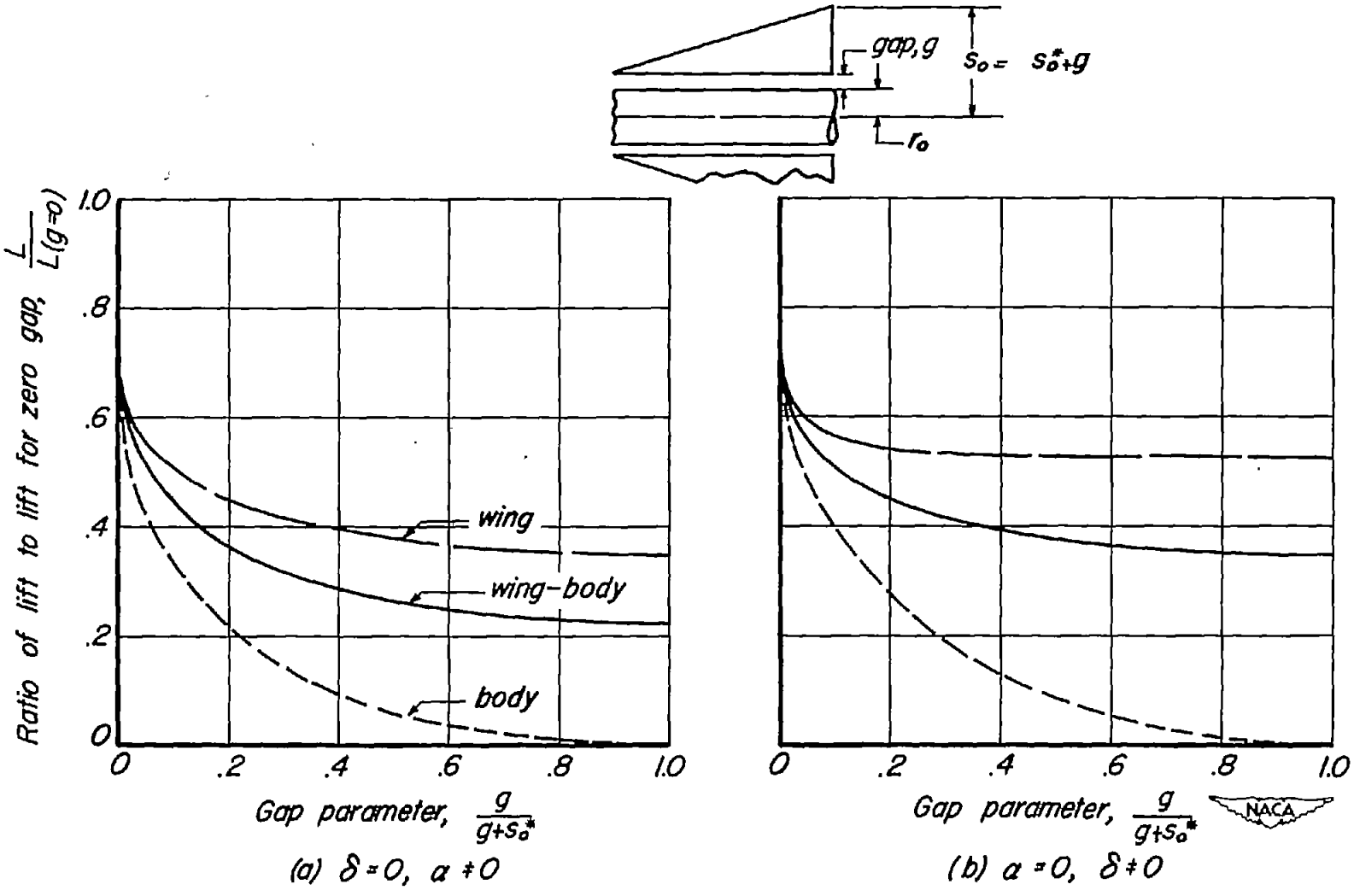



Figure 3.- Variation with gap width of lift of a slender wing-body combination, $\frac{r_o}{s_o^*} = 0.500$

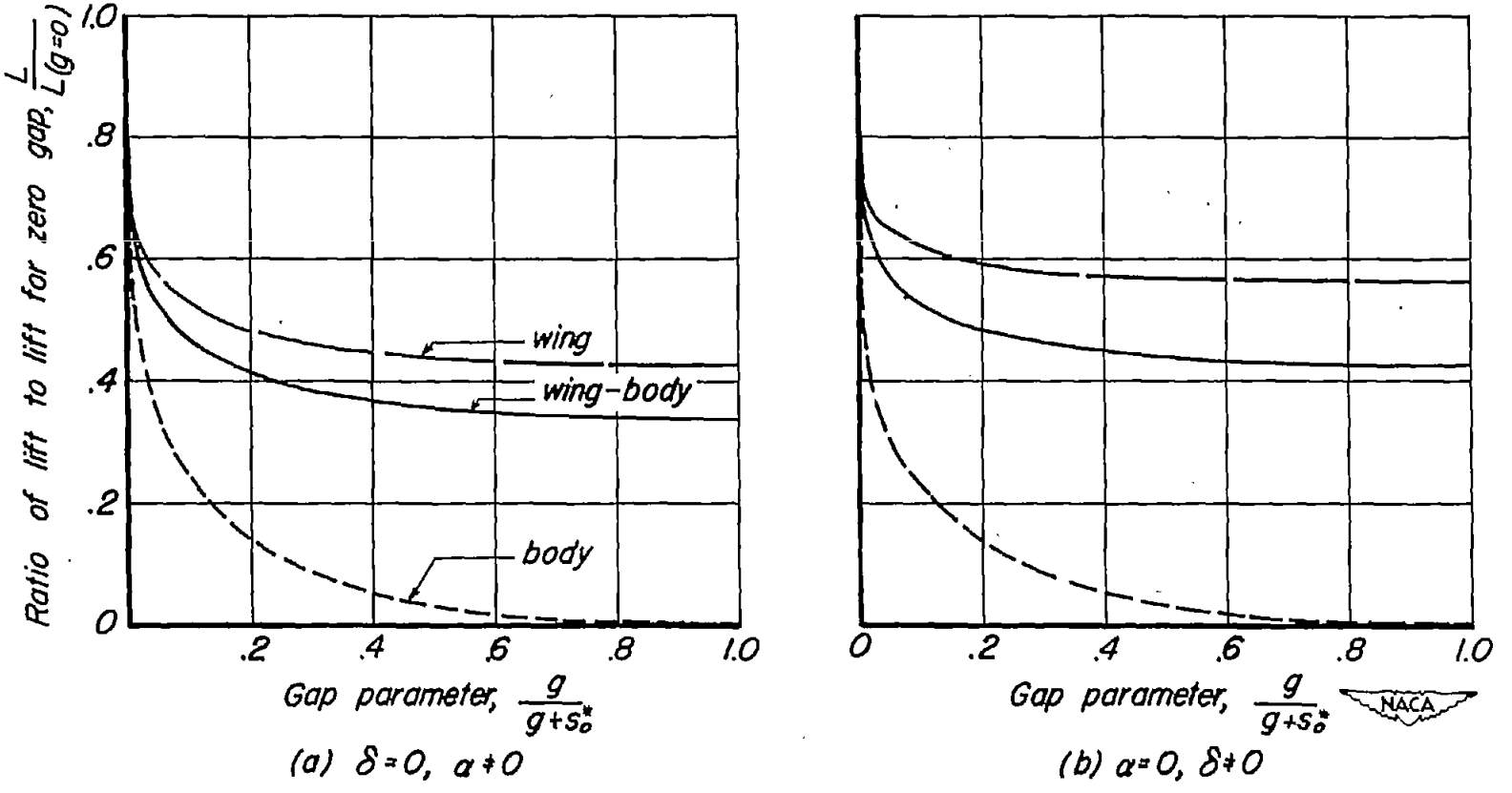
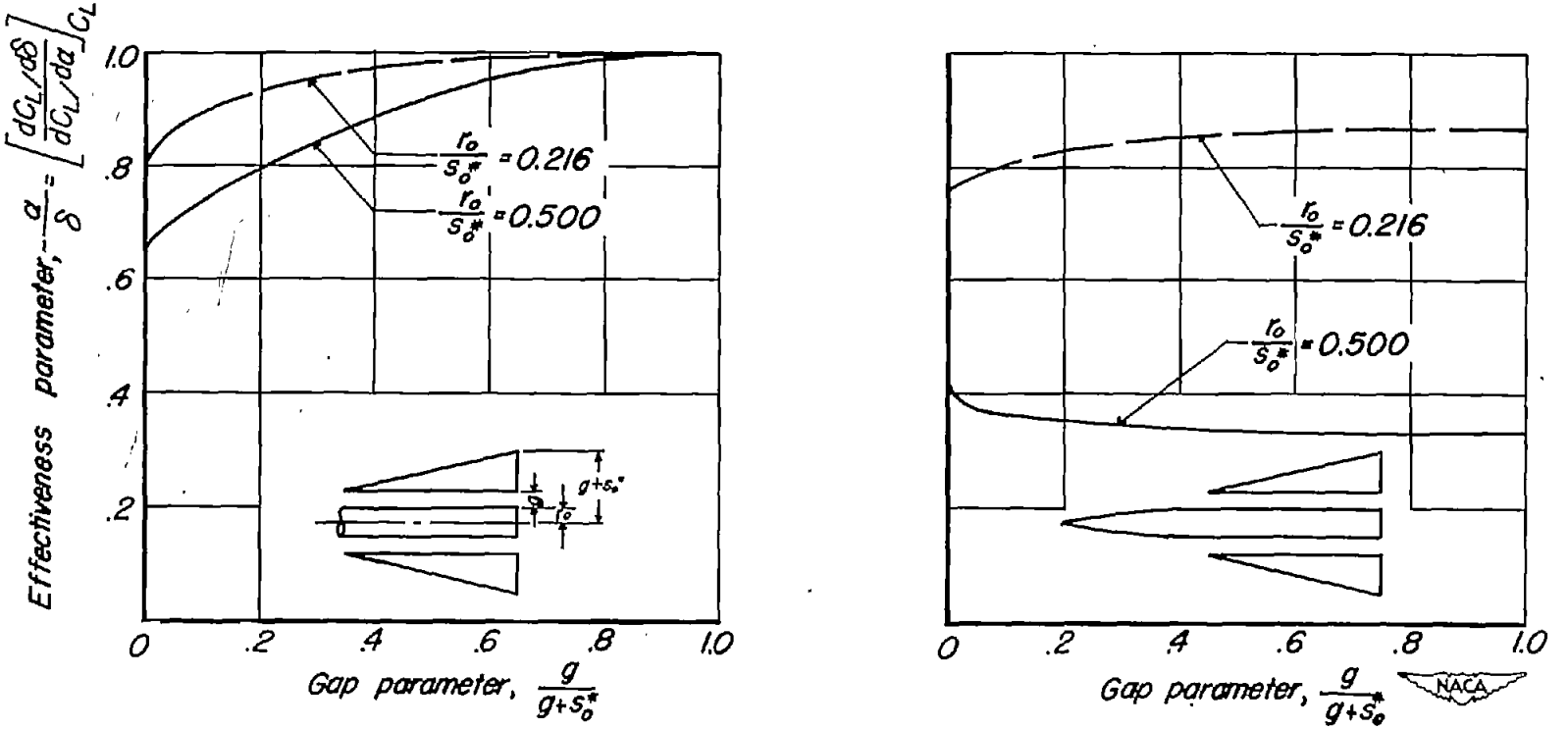



*Figure 4.- Variation with gap width of lift of a slender wing-body combination,* $\frac{r_o}{s_o^*} = 0.216$




(a) Wing-body exclusive of nose & after body

(b) Wing-body + body nose, no after body

*Figure 5.- Variation of effectiveness of all-movable wing of a slender wing-body combination with gap size.*