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MODIFICATION OF WING-SECTION SHAPE TO ASSURE A PREDETERMINED CHANGE IN PRESSURE DISTRIBUTION

By A. Betz

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MODIFICATION OF WING-SECTION SHAPE TO ASSURE A PREDETERMINED CHANGE IN PRESSURE DISTRIBUTION*

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SUMMARY

In order to find an airfoil for a predetermined pressure distribution, the problem must be so posed that the pressure distribution creates no drag. Another fundamental difficulty is that, properly speaking, it is impossible to specify a pressure distribution without first knowing the place where these pressures are to be applied, i.e., the wing-section shape.

This difficulty may be avoided by directing the pressure distribution along the contour of the wing section. Then it becomes possible to define the change in wing-section shape which effects a certain modification of the pressure distribution.

The limitations underlying the required pressure distribution are discussed and it is found that the sole essential limitation is zero drag. The method is illustrated with an example.

The author also refers to several reports on wing sections (references 1 and 2) used in water, i.e., marine propellers and turbines and cavitation phenomena.

INTRODUCTION

Compared to the problem of finding the wing-section shape for a given pressure distribution, the reverse problem of finding the pressure distribution for any wing section is relatively simple and may, in fact, be considered

*"Änderung der Profilform zur Erzielung einer vorgegebenen Änderung der Druckverteilung." Luftfahrtforschung, December 5, 1934, pp. 158-164.

as solved (references 3, 4, 5, and 6). The procedure is to start from an airfoil which can be simply developed into a circle by conformal transformation and thus become readily theoretically tractable. Minor changes then effected on its form will produce, as a rule, only minor changes in the flow which can be calculated by conformal transformation of the original to the new airfoil. Conformal transformation itself is simplified when the form changes are kept small. The best airfoil to start from is a Joukowski airfoil (reference 7) because the trailing edge of the usual Joukowski airfoils ends in a mathematical point (zero edge angle), and so are very thin vicinal to the trailing edge. When the necessary airfoil shape here is substantially thicker, that is, varies considerably from the Joukowski airfoil, one may proceed from the generalized Joukowski airfoils (reference 8) with slightly rounded trailing edge or from Karman-Trefftz airfoils (reference 9) with finite edge angle at the trailing edge.

By virtue of the smallness of the changes, the results are very simple relations between form change and pressure-distribution changes. It therefore suggests the use of these relations for calculating the form change by proceeding from the latter. Here, however, we encounter two fundamental difficulties: First, it is utterly impossible to realize all pressure distributions. A body in a potential flow, as it is assumed, has no drag. Hence, all pressure distributions yielding a drag, must be excluded. The methods of the conformal transformation applied here, stipulate further limitations which, however, may be removed as shown elsewhere. Another difficulty is the fact that a pressure distribution can be specified only when knowing the surface over which the pressure is distributed. But, since our first problem is to find the form of this surface, it is necessary to elucidate the underlying principle.

The pressure distribution is above all important for appraising the processes in the boundary layer, and particularly, of its course along the surface of the body. Hence it is advisable to prescribe the course of the pressure along the development of the airfoil contour. Thereby one is largely independent of the unknown form of the profile, except that the length of the development is for the time not exactly known. However, proceeding from the forward stagnation point, the pressure distribution can be definitely established at least along the greater part of the surface, leaving only an indeterminate zone on the trailing edge due to the uncertain length of the development. On the other hand, since the form changes are minor, the extent of the profile is but little modified, so that the still remaining uncertainty is confined to a very small zone in which, moreover, the pressures are not very changeable.

RELATION BETWEEN CHANGES OF WING-SECTION SHAPE AND PRESSURE DISTRIBUTION

By virtue of Bernoulli's equation

$$p + \rho \frac{v^2}{2} = \text{constant} \tag{1}$$

which expresses the relation between velocity v and pressure p (ρ = fluid density), the pressure distribution is given with the velocity distribution. Thus, in the following we introduce the velocities instead of the pressures.

Let us assume that our original wing section is the result of conformal transformation of a circle in plane z into a plane ζ_1 without modification of the flow at infinity. If the original airfoil is, say, a Joukowski airfoil, then

$$\zeta_1 = z + \frac{a^2}{z} \quad \text{(fig. 1)} \tag{2}$$

The velocities v_1 on the surface of the airfoil are readily computable on the basis of the transformal function.

Then we transform plane ζ_1 to a plane ζ_2 through function

$$\zeta_2 = \zeta_1 + \Delta\zeta \tag{3}$$

whereby $\Delta\zeta = f(\zeta_1)$ and $< < |\zeta_1|$. The velocity assumed as v_1 at a certain point P_1 of plane ζ_1 becomes v_2 in the corresponding point P_2 of plane ζ_2, whereby

$$\frac{v_1}{v_2} = \frac{\partial\zeta_2}{\partial\zeta_1} = 1 + \frac{\partial\Delta\zeta}{\partial\zeta_1} \tag{4}$$

Putting
$$\frac{1}{v_2} = \frac{1}{v_1} + \Delta \frac{1}{v} \tag{5}$$

gives
$$\frac{v_1}{v_2} = 1 + v_1 \Delta \frac{1}{v} \tag{6}$$

From (4) and (6) follows:
$$v_1 \Delta \frac{1}{v} = \frac{\partial \Delta \zeta}{\partial \zeta_1}^{*} \tag{7}$$

consequently
$$\Delta \zeta = \int v_1 \Delta \frac{1}{v} d\zeta_1 \tag{8}$$

So, when the velocity distribution v_1 of the original airfoil and the desired change of these velocities, that is, its reciprocal value $\Delta \frac{1}{v}$ is given, the form changes necessary for the velocity change may be obtained by integration. However, it should be remembered that ζ_1, $\Delta\zeta$, v_1, and $\Delta \frac{1}{v}$ are all directional vectors; that is, complex quantities. Admittedly, v_1 is known in direction and quantity ($\frac{1}{v_1}$ lies in direction of the tangent to the original wing section), whereas of v_2 only the amount can be given because its direction is as yet unknown. And the latter may not be arbitrarily stipulated along with the amount because with a complex function of the plane such as v_2 the amount simultaneously defines the direction. Likewise, the real part defines the imaginary part and vice versa. For this reason the desired data must first be obtained through calculation. (See the following section, page 5.)

Let $R\left(v_1 \Delta \frac{1}{v}\right)$ = real part, and $J\left(v_1 \Delta \frac{1}{v}\right)$ = imaginary part of the complex function $v_1 \Delta \frac{1}{v}$, and $\Delta\xi$ and $\Delta\eta$ the real and the imaginary part (i.e., the ξ and η component) of $\Delta\zeta$, and lastly, $d\zeta_1$ be replaced by its component $d\xi$ and $d\eta$, so that

$$v_1 \Delta \frac{1}{v} = R\left(v_1 \Delta \frac{1}{v}\right) + i\, J\left(v_1 \Delta \frac{1}{v}\right) \tag{9}$$

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*With $\Delta \frac{1}{v}$ sufficiently small compared to $\frac{1}{v_1}$, $v_1 \Delta \frac{1}{v}$ may be replaced by $\frac{\Delta v}{v_1}$, whereby $\Delta v = v_2 - v_1$. But we preserve the more general form $v_1 \Delta \frac{1}{v}$.

$$\Delta\zeta = \Delta\xi + i\,\Delta\eta \tag{10}$$

and

$$d\zeta_1 = d\xi + i\,d\eta \tag{11}$$

Then equation (8) becomes

$$\Delta\xi + i\,\Delta\eta = \int\left[R\left(v_1\,\Delta\,\frac{1}{v}\right) + i\,J\left(v_1\,\Delta\,\frac{1}{v}\right)\right](d\xi + i\,d\eta) \tag{12}$$

The separation of real and imaginary parts give the two real equations

$$\Delta\xi = \int R\left(v_1\,\Delta\,\frac{1}{v}\right)d\xi - J\left(v_1\,\Delta\,\frac{1}{v}\right)d\eta \tag{13}$$

$$\Delta\eta = \int J\left(v_1\,\Delta\,\frac{1}{v}\right)d\xi + R\left(v_1\,\Delta\,\frac{1}{v}\right)d\eta \tag{14}$$

Knowing the function $v_1\,\Delta\,\frac{1}{v}$ along the perimeter of the original wing section, a simple integration along axes ξ and η gives the ξ and η components of the required displacements of the points of the surface.

DEFINITION OF FUNCTION $v_1\,\Delta\,\frac{1}{v}$

Figure 1 shows the original and the derived airfoil. A point P_1 of the original profile (coordinate ζ_1) becomes point P_2 (ζ_2 coordinate) by conformal transformation. The distance $P_1\,P_2$ is $\Delta\zeta$. The trailing edges of both airfoils are assumedly coincident, thus precluding in general a coincidence of the forward stagnation points St_1 and St_2. Since the conformal function is to be regular, point St_1 is transformed again in a stagnation point. Consequently, St_1 and St_2 are points which correspond with each other in conformal transformation.

Now we transform the contours of both wing sections into a straight line (fig. 2). The forward stagnation points are to be neutral points and the distances - as measured along the transformation - from the forward stagnation point of the original and the desired wing section are denoted by s_1 and s_2. The points of the lower surface of the airfoil are figured positive, those of the upper surface, negative. We plot the given velocity v_1 as well as the desired modified velocity v_2 along its surface.

Since the two fields of flow in plane ζ_1 and ζ_2 ζ_2 are created separately by conformal transformation, corresponding points must have equal flow potential Φ. Ascribing zero potential to the forward stagnation points, the potentials existing on the surfaces are found by integrating the velocities, starting from the stagnation point. Identical points in which these have equal potential are then readily recognized. With P_1 and P_2 representing two corresponding points, it is

$$\int_0^{s_1} v_1 \, ds = \int_0^{s_2} v_2 \, ds \tag{15}$$

This immediately yields the still unknown length of development of the desired profile provided, however, that v_2 meets the subsequently discussed presumptions.

Now we can find the point P_2 with velocity v_2 and distance s_2 for each corresponding point of the original wing section with velocity v_1 and distance s_1 from the forward stagnation point, and from the difference of the reciprocal velocities in these corresponding points:

$$\Delta \left|\frac{1}{v}\right| = \left|\frac{1}{v_2}\right| - \left|\frac{1}{v_1}\right| \tag{16}$$

Multiplication with v_1 gives the function with respect to s_1: $\left|v_1\right| \Delta \left|\frac{1}{v}\right| = \left|\frac{v_1}{v_2}\right| - 1$ (17)

Figure 3 (top) illustrates the geometric relation of $\frac{1}{v_1}$, $\frac{1}{v_2}$, and $\Delta\frac{1}{v}$, and (bottom) the connection of 1, $\frac{v_1}{v_2}$ and $v_1 \Delta \frac{1}{v}$ after division by $\frac{1}{v_1}$. It is seen that for $\Delta \frac{1}{v} << \frac{1}{v_1}$ the difference $\left|\frac{v_1}{v_2}\right| - 1$ is almost equal to the projection of $v_1 \Delta \frac{1}{v}$ on the real axis; that is, the real part of $v_1 \Delta \frac{1}{v}$, whence:

$$\left|v_1\right| \Delta \left|\frac{1}{v}\right| = \left|\frac{v_1}{v_2}\right| - 1 \approx R\left(v_1 \Delta \frac{1}{v}\right)$$

From the momentarily given amounts of the initial velocity v_1 and the required velocity v_2, we can accordingly compute the real part of the function $v_1 \Delta \frac{1}{v}$, which, however, must be completed with its corresponding imaginary part $J\left(v_1 \Delta \frac{1}{v}\right)$. This is a problem of the function theory. Accordingly, we transform function $v_1 \Delta \frac{1}{v}$ on plane z, in which the conformal transformation changes the profile contour to a circle with a center such as z_0 and radius r_0. The real parts of the desired function $v_1 \Delta \frac{1}{v}$ are then given on the periphery of this circle.

Assuming the real part of this function at the edge of the circle as the radial component of a flow, the corresponding imaginary part is the tangential component of this flow, which again may be reproduced as field of a source distribution over the edge of the circle. In this case the source strength per unit of length equals the doubled radial component. A source of strength E in point K_2 (fig. 4) with polar coordinates r_0 φ_2 produces in point K_1 (polar coordinates r_0 φ_1) a speed

$$w = \frac{E}{2\, l\, \pi} \tag{18}$$

whereby

$$l = 2\, r_0 \sin\left(\frac{\varphi_2 - \varphi_1}{2}\right)$$

denotes the distance $K_1 K_2$. This speed is in $K_2 K_1$ direction; it forms with the tangent in point K_1 the angle

$$\tau = \frac{\varphi_2 - \varphi_1}{2} \tag{19}$$

The component of w in direction of this tangent is

$$w_t = w \cos \tau = \frac{E}{4\, r_0\, \pi} \cot \frac{\varphi_2 - \varphi_1}{2} \tag{20}$$

For a source distribution of

$$dE = 2R\, r_0\, d\varphi \tag{21}$$

over the edge of the circle, with $R = R\left(v_1 \Delta \frac{1}{v}\right)$

denoting the real part of the function $v_1 \Delta \frac{1}{v}$ plotted over periphery (φ), we have in point P_1 a tangential velocity

$$w_t = \int_0^{2\pi} \frac{a E}{4 r_o \pi} \cot \frac{\varphi - \varphi_1}{2} d\varphi = \frac{1}{2\pi} \int_0^{2\pi} R \cot \frac{\varphi - \varphi_1}{2} d\varphi \qquad (22)$$

The radial component resulting under the influence of all sources graded over the circle edge is zero when the sum of these sources is zero, which is always the case for reasons of continuity. The w_t component is the imaginary part of our $v_1 \Delta \frac{1}{v}$ function at point P_1, which we shall designate with J_1. Accordingly, we can calculate it for every point of the circle and consequently also for the corresponding point of the profile in the z_1 plane by integration.* It is

$$J_1 = \frac{1}{2\pi} \int_0^{2\pi} R \cot \frac{\varphi - \varphi_1}{2} d\varphi \qquad (23)$$

The fact that the integrant for $\varphi = \varphi_1$ becomes infinite, interferes with the evaluation of the integral. But by virtue of

$$\int_0^{2\pi} \cot (\varphi - \varphi_1) \, d\varphi = 0$$

the formula for the imaginary part may also be written as:

$$J_1 = \frac{1}{2\pi} \int_0^{2\pi} (R - R_1) \cot \frac{\varphi - \varphi_1}{2} d\varphi \qquad (24)$$

For $\varphi - \varphi_1$, $R - R_1$ approaches zero in the same measure as $\cot \frac{\varphi - \varphi_1}{2}$ approaches infinity, so that the product remains finite.

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*The imaginary part may equally be computed by resolving the real part in Fourier series and substituting $- \sin n \varphi$ for $\cos n \varphi$; and $\cos n \varphi$ for $\sin n \varphi$. But since this imposes a limitation to a finite number of Fourier series, this method is, as a rule, much more inaccurate than the integration method.

LIMITATIONS IN THE CHOICE OF VELOCITY DISTRIBUTION

The preceding section assumes an altogether arbitrary velocity distribution along the development s_2, which allows the calculation of the $v_1 \Delta \frac{1}{v}$ function and subsequently also the values of $\Delta\zeta$ for each point by integration along the periphery s_1 according to equations (8), (13), and (14). The profiles are permitted to coincide at the trailing edge, so that $\Delta\zeta = 0$. The $\Delta\zeta$ values may be computed along s_1 by starting the integration at the trailing edge. In the end we reach the trailing edge again and compute in general a $\Delta\zeta$ value other than zero. So the new profile no longer closes at the trailing edge. To obtain a closed profile the $\Delta\zeta$ value, resulting with a complete enclosure from (8), must be zero:

$$\oint v_1 \Delta \frac{1}{v} d\zeta_1 = 0 \tag{25}$$

The condition that the closed integral over a function shall be zero assumes a prominent role in the theory of functions. It is met when the function has no residuum within the closed integrating distance, i.e., no ζ_0 point in whose vicinity the function is as $\frac{1}{\zeta - \zeta_0}$. Visualizing the function as speed, such points would denote sources or vortices. Since in any case it is imperative that the function $v_1 \Delta \frac{1}{v}$ displays no singular points outside of the profile and that in addition it disappears at infinity, because the velocity is to remain unchanged at infinity, the function may be expanded according to the powers of $\frac{1}{\zeta_1}$. Then the integration over a closed integration path gives all terms with a power higher than 1 the amount zero, only the term with $\frac{1}{\zeta_1}$ gives a finite amount. Then the premise of zero integral means that in a development of the function according to powers of $\frac{1}{\zeta_1}$ the first term must be absent. Naturally, since the function is to disappear at infinity, no constant term nor positive power of ζ_1 may appear.

The power development may equally be made in the z plane rather than in the ζ_1 plane. With Joukowski and

Karman-Trefftz airfoils* the transformal function of the z on the ζ_1 plane is such that with the transfer of the z to the ζ_1 series, the term with $\frac{1}{z - z_0}$ remains unchanged and goes into that with $\frac{1}{\zeta_1}$, so that the residium is preserved.

We had already transferred the functional values $v_1 \, \Delta \, \frac{1}{v}$ to the periphery in the z plane for computing the imaginary part of $v_1 \, \Delta \, \frac{1}{v}$.

With the power series development

$$v_1 \, \Delta \, \frac{1}{v} = a_0 + i \, b_0 + \frac{a_1 + i \, b_1}{z - z_0} + \frac{a_2 + i \, b_2}{(z - z_0)^2} + \tag{26}$$

we have at the circle periphery $z - z_0 = r_0 \, e^{i\varphi}$, that is,

$$v_1 \, \Delta \, \frac{1}{v} = a_0 + i b_0 + \frac{a_1 + i b_1}{r_0} \, e^{-i\varphi} + \frac{a_2 + i b_2}{r_0} \, e^{-2i\varphi}$$

$$= a_0 + \frac{a_1}{r_0} \cos \varphi + \frac{a_2}{r_0^2} \cos 2 \varphi + \dots$$

$$+ \frac{b_1}{r_0} \sin \varphi + \frac{b_2}{r_0^2} \sin 2 \varphi + \dots$$

$$- i \left[\frac{a_1}{r_0} \sin \varphi + \frac{a_2}{r_0^2} \sin 2 \varphi + \dots \right.$$

$$\left. - b_0 - \frac{b_1}{r_0} \cos \varphi - \frac{b_2}{r_0^2} \cos 2 \varphi - \right] \tag{27}$$

Coefficient b_0 itself becomes zero according to (23) and (24). To find the coefficients a_0, a_1, and b_1

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*The same applies equally to all pertinent transformal functions. The stipulation is the constancy of infinity by the transformation.

of the power series development, we merely define the corresponding coefficients of a Fourier series for the real part of $v_1 \Delta \frac{1}{v}$, that is, $|v_1| \Delta \left|\frac{1}{v}\right|$. It is:

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} |v_1| \Delta \left|\frac{1}{v}\right| d\varphi \\ \frac{a_1}{r_0} &= \frac{1}{\pi} \int_0^{2\pi} |v_1| \Delta \left|\frac{1}{v}\right| \cos\varphi \, d\varphi \\ \frac{b_1}{r_0} &= \frac{1}{\pi} \int_0^{2\pi} |v_1| \Delta \left|\frac{1}{v}\right| \sin\varphi \, d\varphi \end{aligned}\right\} \qquad (28)$$

If the resultant a_0, a_1, and b_1 values are other than zero, the function

$$a_0 + \frac{a_1}{r_0} \cos\varphi + \frac{b_1}{r} \sin\varphi$$

must be subtracted from $v_1 \Delta \frac{1}{v}$, which then voids the particular terms. Of course it is necessary to check whether or not the thus stipulated change in speed v_2 still corresponds to our purposes. In general, this correction entails only a very even change in velocity distribution which in most cases is insignificant since, as a general rule, the purpose is to remove the individual humps or peak values of the velocities which promote premature breakdown of flow. And this occurs primarily through the higher terms of the Fourier series.

We have limited the changes in profile so as to preserve the applicability of conformal transformation. This implies the stipulation that the line integral of the velocity from the forward stagnation point up to the trailing edge of both the top camber Φ_o and the bottom camber Φ_u remain unchanged. In itself this limitation is arbitrary; for instance, it precludes profile changes which correspond to angle-of-attack changes, as in this case it modifies the circulation $\Phi_o - \Phi_u$. However, this may be avoided by providing the original profile with the desired values of Φ_o and Φ_u through other than conformal changes. By modifying the angle of attack, we can, as previously indicated, influence the difference $\Phi_o - \Phi_u$.* A similar enlargement of the profile increases Φ_o and Φ_u in proportion. When combining both methods, any desired value of Φ_o and Φ_u may thus be obtained.

The thought suggests itself as to whether these nonconformal changes rid us of the above premise of $\oint v_1 \, \Delta \frac{1}{v} \, ds = 0$. Since $v_1 \, \Delta \frac{1}{v}$ is an analytical function of ζ_1, without singular points outside of the profile, we likewise may prefer the closed integration path at infinity to that along the profile surface. Then v_1 becomes the constant flow velocity v_∞. The change in velocity Δv_∞ being arbitrarily small at infinity, we have $\Delta v_\infty << v_\infty$. Therefore we may write

$$\oint v_1 \, \Delta \frac{1}{v} \, ds = \oint v_\infty \, \Delta \frac{1}{v_\infty} \, ds_\infty = \oint \frac{\Delta v_\infty}{v_\infty} \, ds_\infty$$

$$= \frac{1}{v_\infty} \oint \Delta v_\infty \, ds_\infty \qquad (29)$$

The real part of $\oint \Delta v_\infty \, ds_\infty$ is the change in circulation around the wing, the imaginary part, the change of the outward flowing fluid resulting from the closed integration path. The circulation may be influenced through angle-of-attack change, and consequently, also the real part of $v_\infty \oint v_1 \, \Delta \frac{1}{v} \, ds$. But, for reasons of continuity, the fluid quantity through the closed integration path must be zero. It can therefore not be influenced, and the result is that

$$J\left(v_\infty \oint v_1 \, \Delta \frac{1}{v} \, ds\right) = 0 \qquad (30)$$

remains the essential limitation of the chosen velocity distribution. The inner reason for this limitation bases on the fact that a velocity distribution, for which

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*Weinig has established simple relationships for changes in velocity distribution with the angle of attack, which are useful for this generalized treatment of the problem (reference 10).

$v_\infty \oint v_1 \Delta \frac{1}{v} ds \neq 0$ yields a change in the force acting on the profile; and specifically the real part signifies the component of this additive force in lift direction, and the imaginary part, that in direction of the drag. Consequently, the premise $(v_\infty \oint v_1 \Delta \frac{1}{v} ds) = 0$ is identical with the originally made stipulation; i.e., that the chosen pressure and velocity distribution must create no drag.

EXAMPLE

The dashed lines in figure 5 show a Joukowski airfoil developed from a circle conformal to

$$\zeta_1 = z + \frac{a^2}{z}$$

The position of the center of the circle z_0 (fig. 1) governs the camber and thickness of the airfoil.

At an angle of attack $\alpha = 7^\circ$ the pressure grading and the local pressure minimum near the nose (fine lines) are as shown in figure 6. Now we attempt to remove this pressure grading and to attain the one shown as full-drawn curve. Since circulation and lift are to be preserved, we must from the very first attempt to equalize the surfaces cut off in the pressure-distribution curve through a corresponding pressure rise at other points, while bearing in mind that this new distribution also is to create no drag. In order to facilitate the choice of a suitable position for the required supplementary surface, we show in figure 7 the prohibited pressure change which corresponds to a

$$v_1 \Delta \frac{1}{v} = \frac{a_1}{r_0} \cos \varphi = - 0.05 \cos \varphi$$

and produces drag. It is readily seen that the equalization of the separated pressure tip must not be effected in the rear portion of the suction side. On the contrary, it must be allocated either to the fore part of the suction side, that is, directly behind the separated tip, or else to the rear portion on the pressure side. We prefer the first and preserve the rear part of the suction side and the whole pressure side as it is.

Figure 8 shows the velocity v_1 over the surface of the Joukowski airfoils plotted against its development.* The forward stagnation point is chosen as null point. Further, the desired change in velocity distribution, the separation of the velocity maximum (v_2) is included. The equally shown potential gradings

$$\Phi_1 = \int_0^s v_1 \, ds \quad \text{and} \quad \Phi_2 = \int_0^s v_2 \, ds$$

were obtained through planimetry of the velocity curves. By virtue of the equalization of the pressure distribution surfaces in a small zone, the potential curves outside of this zone are almost exactly coincident, so that in this case the total length of the development of the upper as well as of the lower surface remains unchanged. Calculating in points with equal potential $(\Phi_1 = \Phi_2)$, the difference of the reciprocal velocities $\Delta \left|\frac{1}{v}\right| = \left|\frac{1}{v_2}\right| - \left|\frac{1}{v_1}\right|$, then plotting these values against the deviation corresponding to v_1 followed by multiplication with v, we finally have the curve $|v_1| \; \Delta \left|\frac{1}{v}\right|$ which is also shown in figure 8. There are no finite values except in the vicinity of the effected pressure changes. The transfer of the latter values to the transformation of the circle which corresponds to the Joukowski airfoil (fig. 1), gives the heavy curve of figure 9. At this point we must ascertain whether the desired velocity distribution is serviceable. Planimetration gives

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} |v_1| \; \Delta \left|\frac{1}{v}\right| d\varphi = 0.0016$$

The constant term a_0 is best removed by effecting a change on the pressure side (thin line in fig. 9), which at the same time assures a somewhat smoother shape of the curve. For this modified curve we determine

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*The length unit chosen for this and the following graphs is the distance of the trailing edge of the airfoil from the null point of the ζ_1 and the ζ_2 plane. This is equal to 2a, whereby a has the notation given in (2) and figure 1. The velocities are made nondimensional through division with the velocity at infinity V_∞.

$$\frac{a_1}{r_o} = \frac{1}{\pi} \int_0^{2\pi} |v_1| \ \Delta \left|\frac{1}{v}\right| \cos \varphi \ d\varphi = - 0.0053$$

$$\frac{b_1}{r_o} = \frac{1}{\pi} \int_0^{2\pi} |v_1| \ \Delta \left|\frac{1}{v}\right| \sin \varphi \ d\varphi = - 0.0012$$

The subtraction of the values $\frac{a_1}{r_o} \cos \varphi + \frac{b_1}{r_o} \sin \varphi =$ $-0.0053 \cos \varphi -0.0012 \sin \varphi$ from the curve $R\left(v_1 \Delta \frac{1}{v}\right)$, gives the difference shown as dashed line. The corresponding changes of the curves in figure 6 are also shown as dashed lines. These corrections are ostensibly minor and do not detract from the intended purpose.*

With the corrected values (dashed curve) for $|v_1| \ \Delta \left|\frac{1}{v}\right| = R\left(v_1 \Delta \frac{1}{v}\right)$, we then compute the imaginary values $J\left(v_1 \Delta \frac{1}{v}\right)$ for different points of the circle periphery according to (24) by plotting and planimetry of the corresponding integrants. The result is shown in figure 9 (top). With this function the first terms of the Fourier expansion must of themselves be zero. But owing to the inevitable inaccuracies, it is advisable to check this characteristic particularly and, if necessary, make minor corrections on the shape of the curve.

The points of the contour of the Joukowski airfoil projected on the ξ axis, then on the η axis, followed by plotting of $R\left(v_1 \Delta \frac{1}{v}\right)$ as well as $J\left(v_1 \Delta \frac{1}{v}\right)$ against both projections (figs. 10 and 11) gives the coordinate displacements $\Delta\xi$ and $\Delta\eta$ according to (13) and (14) through progressive planimetry.

For the trailing edge, to which we return after integrating over the whole periphery, we must have $\Delta\xi = \Delta\eta = 0$. This condition is an exceptional criterion for the correct integration. Such a criterion is, in fact, needed, because the pertinent surfaces consist, as seen from

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*Moreover, part of this correction could have been avoided by changing the angle of attack of the original airfoil. But we shall keep to the request that the new wing section shall have the same lift as the original airfoil.

figures 10 and 11, of large positive and negative parts, so that the result, as comparatively minor discrepancies in positive and negative component parts, is quite apt to become inaccurate. Since the terms for $\Delta\xi$ and $\Delta\eta$ consist of two integrals, it is not readily ascertainable how the inaccuracies are distributed over the individual integrals, if the cited criterion is not correct. Yet for obviating the inaccuracies, it is important to know them individually for every integral. This may be accomplished in the following manner: The ordinates ξ and η of the original airfoil may be represented as functions of the circle perimeter of plane z; that is, as functions of angle φ. Then ξ and η may be split into

$$\xi = \xi_1 + \xi_2 \qquad \eta = \eta_1 + \eta_2 \tag{31}$$

whereby

$$\xi_1 = \left(r_0 + \frac{a^2}{r_0}\right) \cos\varphi \quad \text{and} \quad \eta_1 = \left(r_0 - \frac{a^2}{r_0}\right) \sin\varphi \tag{32}$$

As the terms for $R\left(v_1 \Delta \frac{1}{v}\right)$ and $J\left(v_1 \Delta \frac{1}{v}\right)$ may contain no terms with $\sin\varphi$ and $\cos\varphi$, we have:

$$\oint R\, d\xi_1 = \oint J\, d\xi_1 = \oint R\, d\eta_1 = \oint J\, d\eta_1 = 0 \tag{33}$$

Thus it is possible to check each one of these four integrals to their being zero and, if necessary, to remove the discrepancies by minor changes of the limitedly exact curves for $R\left(v_1 \Delta \frac{1}{v}\right)$ and $J\left(v_1 \Delta \frac{1}{v}\right)$.

As concerns the lack of individual check on the remaining integrations over ξ_2 and η_2, these are usually so small compared to the separated integrals over ξ_1 and η_1, that any existing discrepancy is no longer of any significance. Figures 12 to 15 illustrate the eight functions to be integrated, which manifest the subordinate importance of the integrations over ξ_2 and η_2 (the scale used for ξ_2 is greater than for ξ_1). The ensuing values for $\Delta\xi$ and $\Delta\eta$ are compiled in figure 16. The resultant changes of the wing section are included in figure 5. The full line in this figure is the original; the dashed line the modified shape.

Translation by J. Vanier,
National Advisory Committee for Aeronautics.

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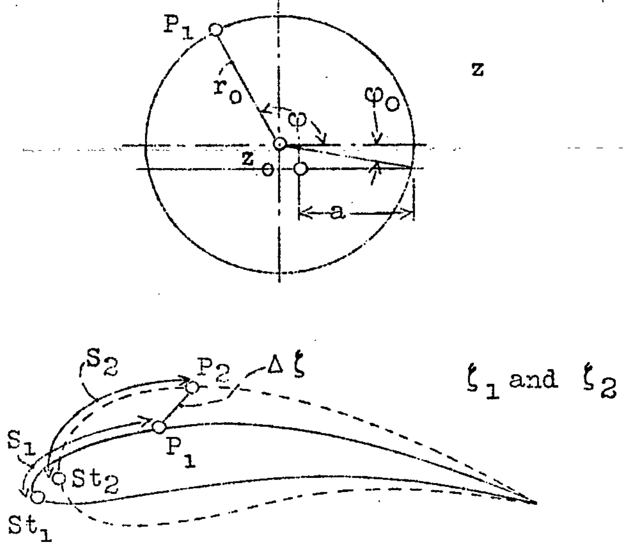

Figure 1.-Transformation of a circle (z plane) in a Joukowsky airfoil (ζ_1 plane, full) and a modified airfoil (ζ_2 plane, dashed).

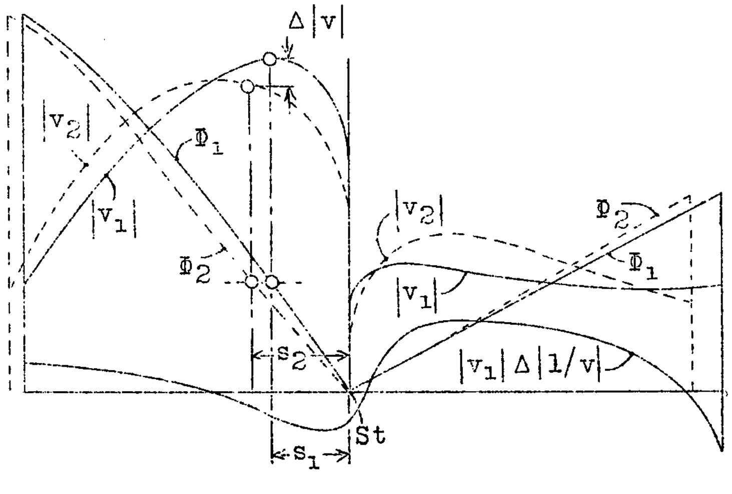

Figure 2.-Course of velocities and potentials along the development of the profiles.

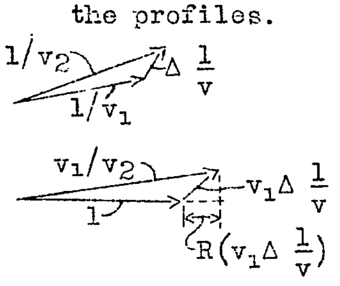

Figure 3.-Geometrical relationship of the different velocity vectors.

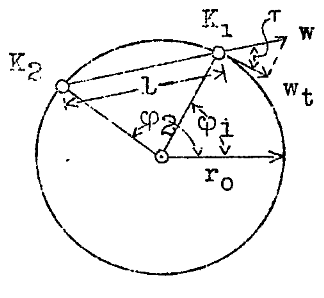

Figure 4.-Illustration for computing the imaginary from the real part.

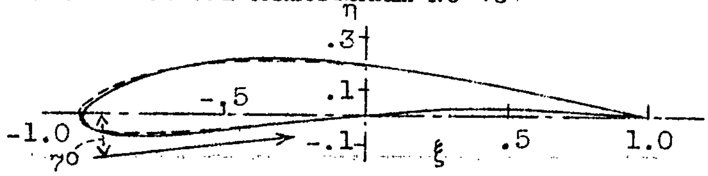

Figure 5.
Original airfoil ——
Modified profile ---

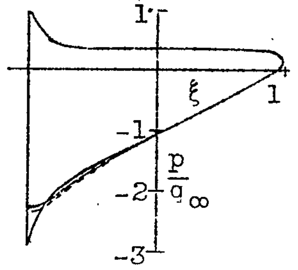

Figure 6.

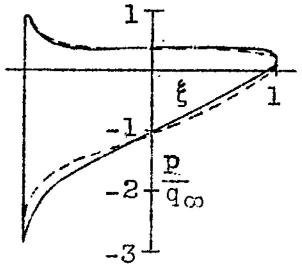

Figure 7.

Figure 6.- Pressure distribution change of original profile.

Figure 7.- Prohibited pressure distribution change.

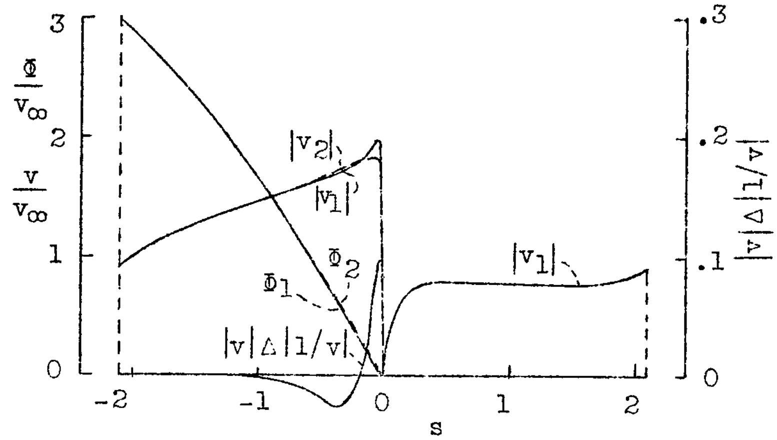

Figure 8.- Course of velocity and potential along the development of the profile.

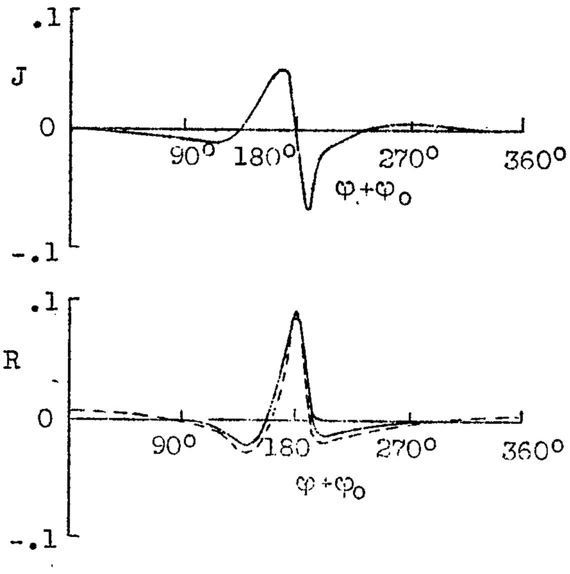

Figure 9.- Course of $R(v_1\Delta\ 1/v)$ and $J(v_1\Delta\ 1/v)$ along the circle periphery in plane z.

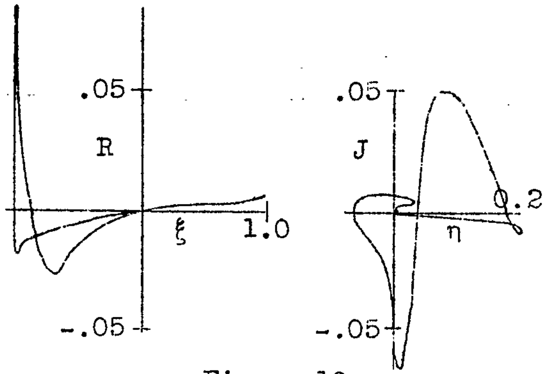

Figure 10.

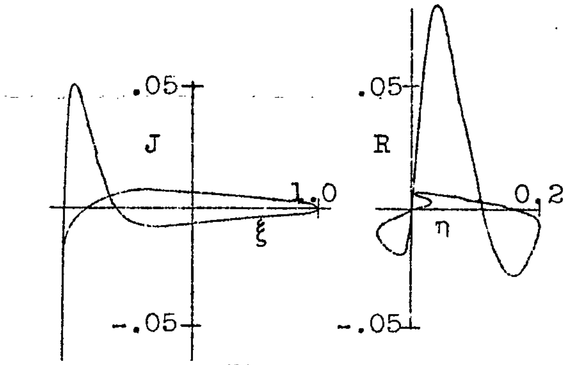

Figure 11.

Figures 10,11.-Course of R and J along the profile contour projected on axis ξ and on axis η.

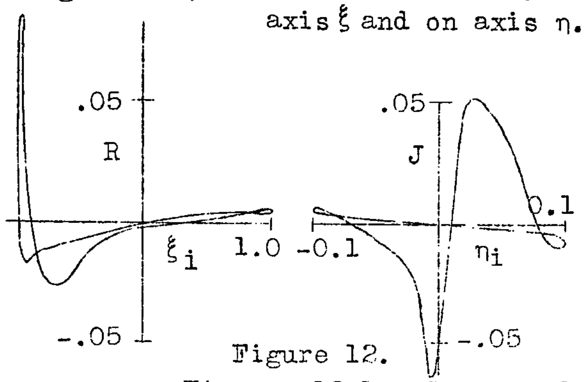

Figure 12.

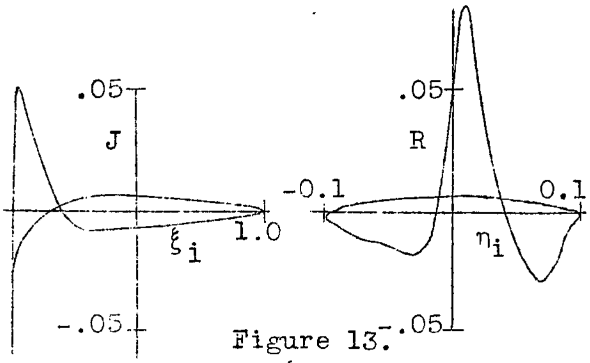

Figure 13.

Figures 12,13.-Course of R and J along axes ξ_i and η_i.

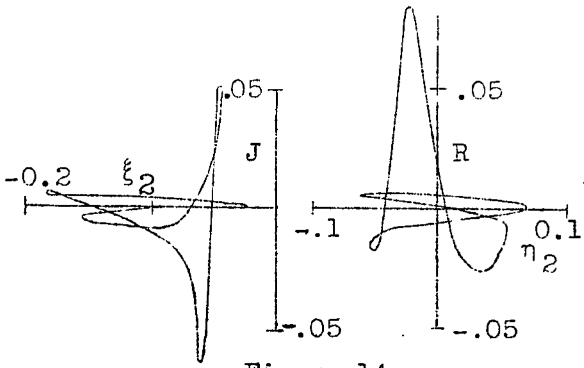

Figure 14.

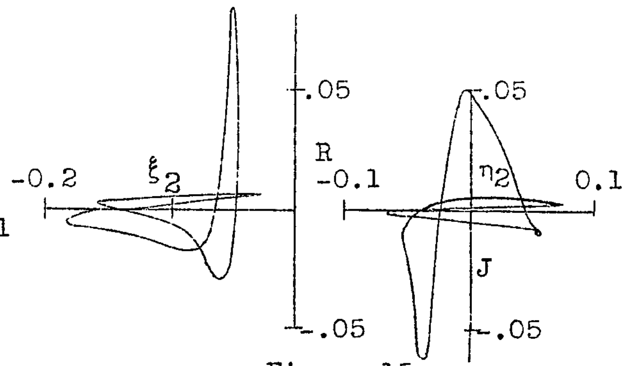

Figure 15.

Figures 14,15.-Course of R and J along axes ξ_2 and η_2.

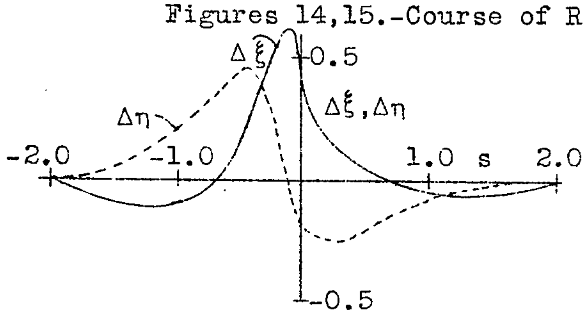

Figure 16.-The resultant displacements Δξ and Δη of the profile points.