

Geopotential & Geometric Altitude

The standard atmosphere is defined in terms of geopotential altitude. The idea behind this concept is that a small change in geopotential altitude will make the same change in gravitational potential energy as the geometric altitude at sea level. Mathematically, this is expressed as $gdZ = GdH$ where H stands for geopotential altitude and Z stands for geometric altitude, g is the acceleration of gravity and G is the value of g at sea level. The value of g varies with altitude and is shown in elementary physics texts to vary as

$$\frac{g}{G} = \left(\frac{E}{Z + E} \right)^2 \quad (1)$$

where E is the radius of the earth. So,

$$dH = \frac{g}{G} dZ = \left(\frac{E}{Z + E} \right)^2 dZ \quad (2)$$

and integrating yields

$$\int_0^H dH = \int_0^Z \left(\frac{E}{Z + E} \right)^2 dZ \quad (3)$$

$$H = \frac{EZ}{Z + E} \quad (4)$$

$$Z = \frac{EH}{E - H} \quad (5)$$

Observe that $0 \leq H \leq Z$ and that $H = Z$ only when $Z = 0$. Also, $H < E$ and as Z increases, so does H and in the limit, $H \rightarrow E$.

While Z and H are virtually identical at low altitudes, you can calculate that $Z = 86$ km corresponds to $H = 84.852$ km. (Use 6356.75 km for the radius of the earth). At this altitude, g is 0.9735 times the value at sea level. If you don't like the definition of H as a differential, you can regard $H = EZ/(Z + E)$ as the definition of H and then derive $dH/dZ = g/G$.

The Perfect Gas Law

The equation of state of a perfect gas is

$$\rho = \frac{MP}{RT} \quad (6)$$

where P is the atmospheric pressure, R is the universal gas constant, ρ is the density, T is the absolute temperature and M is the mean molecular weight of air. M is assumed constant (=28.9644) up to 86 km where dissociation and diffusive separation become significant. R is 8.31432 joules $K^{-1}\text{mol}^{-1}$.

The Hydrostatic Equations

The fundamental hydrostatic equation is

$$dP = -\rho g dZ = -\rho G dH \quad (7)$$

and using the perfect gas law, this becomes

$$dP = -\frac{MP}{RT} G dH \quad (8)$$

This equation leads directly to the calculation of pressure in the standard atmosphere. The temperature in the standard atmosphere is assumed to be constant or linearly varying in each of the seven layers defined by the COESA document. Within an atmospheric layer, the temperature T is a linear function of the geopotential altitude H .

$$T = T_b + L(H - H_b) \quad (9)$$

where L is the constant gradient of temperature and T_b and H_b are the temperature and geopotential altitude at the base of the layer. The hydrostatic equation then becomes

$$dP = -\frac{MG}{R} \frac{P}{(T_b + L(H - H_b))} dH \quad (10)$$

and the pressure at any value of H within this layer is found by integration of this equation

$$\int_{P_b}^P \frac{dP}{P} = - \int_{H_b}^H \frac{MG}{R(T_b + L(H - H_b))} dH \quad (11)$$

The right hand integral takes different forms, depending upon whether L is zero or not. When $L = 0$, the integral is

$$\ln \left(\frac{P}{P_b} \right) = -\frac{GM}{RT_b} (H - H_b) \quad (12)$$

and when L is not zero, the integral is

$$\ln \left(\frac{P}{P_b} \right) = -\frac{GM}{RL} \ln \left(\frac{T_b + L(H - H_b)}{T_b} \right) \quad (13)$$

Writing these equations in exponential form, when $L = 0$

$$\frac{P}{P_b} = \exp \left(-\frac{GM}{T_b R} (H - H_b) \right) \quad (14)$$

and

$$\frac{P}{P_b} = \left(\frac{T_b + L(H - H_b)}{T_b} \right)^{-\frac{GM}{RL}} \quad (15)$$

when L is not zero.

You can see now why geopotential altitude is used for the definition of the standard atmosphere. If Z were used, then g would appear in the equations instead of G and the variation of g with altitude would have to be included in the integration, making a rather complicated equation.

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