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LINEARIZED SUPERSONIC THEORY OF CONICAL WINGS

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PREFACE

The work on the problems in this report was started by the author at Douglas Aircraft, Santa Monica, in January, 1946. The main bulk of the work was done at the Jet Propulsion Laboratory, California Institute of Technology.

Many of the methods and results of this report have been presented at seminars, in particular at Johns Hopkins University and California Institute of Technology. Some of the formulas obtained have been used in References 1, 2, and 3.

The main concepts in the theory of conical wings are drawn from the work of A. Busemann (Cf. Ref. 4). Through lectures and discussions the author also received many valuable ideas from W. D. Hayes, R. T. Jones, and in particular H. J. Stewart. Some of the results in Sections I, II, III, and IV were obtained independently by the author and others, in particular W. D. Hayes, and also by many research workers here and abroad working with widely different methods. Sections V and VI are believed to be essentially new, as regards both the basic solutions and the proposed application of these solutions.

Miss Martha E. Graham of the Douglas Aerodynamics Department carried out many special computations and supplied valuable criticism. At the Jet Propulsion Laboratory, George Morikawa worked on the manyscript and H. J. Stewart read it critically.

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INTRODUCTION AND SUMMARY

The theory of conical flow was initiated by Busemann in Reference and developed further in Reference 4 where the foundations of a theory of conical wings are laid. However, these references contain only a brief outline of the methods and the solutions for only two special cases of conical wings. Some powerful methods were introduced but often presente with insufficient explanations. The present report* tries to clarify these methods and develop them further. It also treats a large number of cases not considered by Busemann. The emphasis is on a detailed treatment of the general methods and of the basic conical solutions. At the same time some new ways of using conical solutions in applied wing theory are pointed out.

Section I is devoted to the foundations of conical theory. Boundary conditions are discussed in detail, as well as general methods like the oblique transformation. Formulas are derived which will simplify the work in the later sections.

Sections II to VI, inclusive, are devoted to detailed studies of various basic conical solutions. In general, expressions for the entire flow field are derived. In the simplest application of conical theory one divides a given wing of finite chord and span into various regions, in each of which some conical solution is valid. The effect of angle of attack is also separated from the effect of thickness with the aid of the superposition

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principle. Basic solutions useful for such purposes are contained in Sections II and III. With the aid of these solutions the flow field near wings of simple planforms and profile may be determined. This is the way conical theory is customarily applied (Cf. Refs. 1, 6, and 7). However, the use of conical flow theory goes far beyond this obvious application. One further method of application was indicated by Busemann, namely, a method of determining flow conditions (downwash and sidewash) behind a three-dimensional wing. The method uses wings of constant-lift distribution. The solutions for such wings are worked out in Section IV. Reference 2 contains some applications to wings of practical interest where the method is illustrated in detail.

Another type of wing, which to the author's knowledge has not been considered before, is the mixed type consisting of a region of constant lift adjacent to a flat plate. The solutions are easily obtained with the methods developed in the present report. Essentially these are solutions of a certain type of interference problem. With their aid the lift problem may be solved for a much larger class of planforms than was possible with the solutions from Section III. In particular, wings with low aspect ratio may be treated, as pointed out in Section III-K.

In Section VI a different type of interference problem is studied. The wings considered consist of adjacent flat regions at different angles of attack, and have many applications. The application to control surfaces has been worked out in Reference 3.

At the time the essential part of the work in the present report was done, conical methods frequently provided the fastest and sometimes the only way of solving many problems in applied wing theory (Cf. Refs. 1, 2, and 3). Although other methods have been developed since, it is believed that the solutions presented in the present report still have considerable practical application. The method of superimposing conical flow fields also has the advantage of being very intuitive. Furthermore, the theory of conical flow is of considerable theoretical interest. In the Busemann theory, the problem of finding the flow around a conical wing is reduced to that of finding certain analytic functions of a complex variable. With the aid of the powerful methods of the theory of analytic functions, the essential structure of the solution may often be seen directly from the boundary conditions. In this way a deeper understanding of the nature of conical flow fields is provided. It is to be hoped that eventually similar results may be obtained for more general flow fields.

I. GENERAL THEORY OF LINEARIZED SUPERSONIC FLOW*

In this section some fundamental properties of linearized supersonic flow will be studied. In particular, the concepts of conical flow and conical wings will be introduced. The main purpose, however, is to furnish the necessary basis for the study of specific conical wings in the remainder of the report.

A. Fundamental Equations and Boundary Conditions

In general, it is assumed in this report that the free stream Mach number has the value $M = \sqrt{2}$. As will be shown in Section I-M, a problem at a different supersonic Mach number may be reduced to an equivalent

*The nomenclature used in this report is given in Table I.

problem at $M = \sqrt{2}$ by a Prandtl-Glauert transformation. Although this reduction is not universally valid, it is admissible in the linearized theory for the type of bodies (wings) treated in the present report. The free stream velocity will be denoted by w_∞, taken in the direction x_3.

The fundamental equations for linearized supersonic flow at $M = \sqrt{2}$ are

$$g_{x_1 x_1} + g_{x_2 x_2} - g_{x_3 x_3} = 0 \quad (g = u,\ v,\ w \text{ or } \Phi) \tag{1.1}$$

and the equations of irrotationality interconnecting the various perturbation velocities and the velocity potential are

$$\operatorname{grad} \Phi = (u,\ v,\ w) \tag{1.2a}$$

$$\operatorname{curl} \ (u,\ v,\ w) = 0 \tag{1.2b}$$

Here (u, v, w) denotes a vector (the perturbation velocity) with three components u, v, and w.

If the free stream Mach number is M, Equation (1.1) will have to be replaced by

$$g_{x_1 x_1} + g_{x_2 x_2} - \frac{1}{m^2} g_{x_3 x_3} = 0, \quad m = \frac{1}{\sqrt{M^2 - 1}} \tag{1.1'}$$

The pressure and the acceleration potential (Cf. Ref. 8) are obtained from the perturbation velocity w by

$$p - p_\infty = -\rho_\infty \quad w_\infty w \tag{1.3a}$$

$$\text{acceleration potential} = w_\infty \ w \tag{1.3b}$$

When there is no danger of misunderstanding, the component w will be referred to as pressure.

The boundary conditions usually considered are:

On the surface of a body in a supersonic stream the flow vector (u, v, $w_\infty + w$) is tangent to the body. (1.4)

At a great distance upstream of the body (infinity), the velocity vector and the thermodynamic quantities have the constant values (0, 0, w_∞), p_∞, ρ_∞, etc. (1.5)

In the problems treated in the present report, condition (1.5) still applies. But condition (1.4) will be modified for several reasons: First, it presupposes that the shape of the body is given. In some problems, however, the pressure will be given instead. In these cases condition (1.4) will be replaced by a suitable condition on w. Second, when the shape of the given body is of a certain type, condition (1.4) may be simplified and stated as a condition on v only. Third, in treating conical flow some important mathematical transformations of the boundary conditions will be made which, however, will not change their content or introduce any new approximations. In some cases it will also be necessary to add some further conditions in order to determine the solutions uniquely.

These modifications of the boundary conditions will be discussed in Sections I-B and I-K.

B. Wings (Planar Systems), Symmetry Properties, and Boundary Conditions

All specific solutions in Sections II to VI, inclusive, will be for conical wings as defined in Section I-K. The fundamental properties discussed in Section I-B will apply to conical wings as well as to more general wings; in particular, a large class of finite three-dimensional wings as actually used on airplanes or missiles.

In linearized theory it is often permissible to apply the boundary condition (1.4), or some substitute therefor, to a region which is contained in the X_1X_3-plane. Such a body will be called planar, and in this case the X_1X_3-plane will be referred to as the plane of the wing. Of course the wing is not contained in this plane except for the trivial case of a flat plate at zero angle of attack. Prescribing a quantity on the wing will then mean prescribing it on the vertical projection of the wing on the X_1X_3-plane.

The simplest case is that of a wing whose shape is prescribed in such a way as to satisfy the following condition:

The tangent plane is everywhere almost horizontal, and the wing has some points in common with the horizontal X_1X_3-plane. (1.6)

For such a wing one defines:

Local angle of attack at the point $P = (x_1, x_2, x_3)$ is the angle between the tangent plane at P and the X_1X_3-plane measured in a plane x_1 = constant. (1.7)

For a wing satisfying condition (1.6), condition (1.4) is replaced by the following

If the local angle of attack at $P = (x_1, x_2, x_3)$ is α, then $v = -w_\infty \alpha$ at $\bar{P} = (x_1, 0, x_3)$. (1.4')

Note that here a condition at a point P on the wing is replaced by a condition at its projection $\overline{P}$ in the plane of the wing. Condition (1.4') may be expressed as v = known function of x_1 and x_3 in vertical projection of wing on X_1X_3-plane.

When the pressure or the lift is prescribed, condition (1.4) will be replaced by a condition on w; and in all cases of interest in the present report the values of w will be given in the plane of the wing. Then condition (1.4) will be replaced by:

w = known function of (x_1, x_3) in a region which is
the projection of the wing on the X_1X_3-plane. (1.4")

Finally, certain mixed types of wing are of importance. Here v is given over one part of the wing and w over a different part.

Actually, the region in the X_1X_3-plane may have a top side and a bottom side. Points on the top side will be considered as projections of points on the top surface of the wing. Their x_2-coordinate will be denoted by 0+. Similarly $x_2 = 0-$ for points on the bottom side. The value of a function f at $(x_1, 0+, x_3)$ will be considered as the limiting value

$$\lim_{\delta \to 0, \delta > 0} f(x_1, \delta, x_3)$$

In the same way $f(x_1, 0-, x_3)$ is defined as

$$\lim_{\delta \to 0, \delta > 0} f(x_1, -\delta, x_3)$$

Very often one of the velocity components will have different values for $x_2 = 0+$ and $x_2 = 0-$. In particular, such a discontinuity may be prescribed in the boundary conditions (1.4') or (1.4''). Discontinuities may also occur in the X_1X_3-plane off the wing.

In general, condition (1.5) plus a prescription of the values of v or w on the wing will not determine the flow field uniquely. One will need conditions on the part of the X_1X_3-plane off the wing and sometimes some conditions on the behavior at the edges.

For a study of the first questions some observations are needed about symmetry properties of the flow and discontinuities in the X_1X_3-plane. Proofs of some of the assertions made later in Section I-B would have to be based on the invariance of the Equations (1.1) and (1.2) under reflection (Cf. Sections I-C and I-D).

When the word symmetry is used without specification, it refers to symmetry with respect to the X_1X_3-plane since for the present purpose this is the important case.

A function $f(x_1, x_2, x_3)$ is said to be symmetric, with respect to the X_1X_3-plane, if it is even in x_2; i.e., if

$$f(x_1, -x_2, x_3) = f(x_1, x_2, x_3) \tag{1.8a}$$

It is antisymmetric, with respect to the X_1X_3-plane, if it is odd in x_2; i.e., if

$$f(x_1, -x_2, x_3) = -f(x_1, x_2, x_3) \tag{1.8b}$$

If $P = (x_1, x_2, x_3)$, then $(x_1, -x_2, x_3)$ is called its reflected point P_r.

The following simple property is fundamental:

If f is antisymmetric, it is either zero or discontinuous for $x_2 = 0$. (1.9)

In proof of this statement, let

$$f(x_1, 0+, x_3) = \lim_{\delta \to 0, \delta > 0} f(x_1, \delta, x_3) = +a$$

Then, by condition (1.8b)

$$f(x_1, 0-, x_3) = \lim_{\delta \to 0, \delta > 0} f(x_1, -\delta, x_3) = -a$$

Obviously, $a \neq 0$ means a discontinuity of magnitude 2a. With respect to symmetry properties there will be two main types of body shapes.

Symmetrical nonlifting wings. Such wings are bisected by the X_1X_3-plane. Whenever a point P is on the upper surface, its reflected point P_r is on the lower surface. According to conditions (1.4′) and (1.8b), a discontinuous antisymmetric distribution of v is prescribed on the wing (i.e., on its projection in the X_1X_3-plane).

Lifting wings of zero thickness. Here the prescribed distribution of v is continuous and symmetric.

Corresponding to these two types of wings there are two types of flow.

These two types will be denoted, rather arbitrarily, as symmetric and lifting. The properties are summarized in the following tabulation in which S = symmetric and A = antisymmetric:

$$
\left.\begin{array}{llllll}
\text{Case} & \Phi & u & v & w & \text{body} \\
\text{Symmetric} & S & S & A & S & \text{symmetric} \\
\text{Lifting} & A & A & S & A & \text{zero thickness}
\end{array}\right\} \qquad (1.10)
$$

The importance of these two types of flow is that any flow may be decomposed into a sum of the two types of flow, for Φ may be represented as a sum of an odd and an even function, and the behavior of Φ determines the other quantities uniquely by condition (1.2a).

For this reason all wings considered later in the present report will be either symmetrical and nonlifting or lifting and of zero thickness.

The fundamental principle for determining the boundary conditions of the wing is:

The X_1X_3-plane may carry a discontinuity in v or w only where there is a wing. (1.11)

Combining condition (1.11) with conditions (1.9) and (1.10), one obtains:

For a symmetrical wing $v = 0$ in the X_1X_3-plane off the wing. (1.12a)

For a lifting wing of zero thickness $w = 0$ in the X_1X_3-plane off the wing. (1.12b)

Note that a discontinuity in u is possible off the wing. This discontinuity is nothing but a vortex sheet.

Reference 2 contains a detailed discussion of some of the principles which have just been mentioned. They are actually of great importance for supersonic wing theory for problems of pressure distribution in an induced flow field behind a finite wing. They will also be used consistently in the present report both in the derivation of the basic solutions and in Sections IV-B, V-E, and VI-E.

Conditions (1.12) and (1.4) prescribe either v or w over the entire X_1X_3-plane. There may then occur two types of boundary value problems which are essentially different from a mathematical point of view. Take a symmetrical wing with given shape, say, of constant angle of attack. Then $v = \pm v_o$ on the wing and $v = 0$ off the wing in the X_1X_3-plane. Hence v alone is prescribed. This problem is essentially easy. Although Busemann's conical methods may furnish rapid and elegant solutions (Cf. Section II), there are other straightforward solutions, e.g., by a source sheet which may be determined directly from the shape of the body. Similarly in the lifting case if the lift is prescribed, $w = \pm f(x_1,x_3)$ on the wing where f is a known function and $w = 0$ off the wing. Again this is a straightforward problem. The solution may, e.g., be had from the acceleration potential (Cf. Ref. 8) which is directly given from the lift distribution. However, conical methods may again be used to great advantage when the planform is conical (Cf. Section IV). Finally it should be noted that since v and w satisfy the same equation, any solution for a symmetrical wing of prescribed shape yields a solution for a wing of zero thickness and prescribed lift distribution if v is replaced by w.

However, if a lifting wing of prescribed shape is given, the problem is much more difficult. On the wing v is prescribed; off the wing w is known (= 0). Most theories of a lifting surface have led to an integral equation which is very easy to set up but for whose solution there exist no straightforward methods (Cf. Refs. 9 and 10). In the solution of such a problem Busemann's methods show their greatest power. The lift problem is reduced to one in classical potential theory which may be solved by the powerful tools of theory of complex variables. Ways by which the mixed boundary value problem may be reduced to well-known problems will be discussed in Section I-K. Of course, Busemann's method applies directly to conical wings only, but one may obtain a very large class of finite wings by superposition of conical wings (Cf. Refs. 1, 2, and 3 or Section V-E).

Finally for a mixed type of wing (Cf. Section V) the situation may be even more complicated. For the lifting case v is prescribed on part of the wing, w is prescribed on some other part, and w = 0 off the wing. Again Busemann's concept will provide easy solutions.

In some applications the addition of condition (1.12) is not sufficient to make the solution unique, but it is necessary to prescribe the behavior of the solution at the edges. First, some definitions are needed. An edge is called supersonic if its sweepback angle is less than the complement of the Mach angle; i.e., if the projection of the free stream velocity w_∞ on a plane normal to the edge is still supersonic. If this projection is subsonic or zero, the edge is said to be subsonic. Edges may also be classified as leading and trailing. If a line in the x_3-direction leaves the wing when crossing an edge, the edge is said to be trailing; in the opposite case it is leading. In the intermediate case when the edge, or rather its projection on the X_1X_3-plane, is parallel to the x_3-direction, it is called an unyawed side edge. The essential assumption about the edges is:

The behavior of the flow at a subsonic edge is qualitatively the same as in incompressible flow. (1.13)

This rather vague principle will now be amplified. Consider a leading edge. If it is supersonic, conditions at the edge are easily determined from the geometry of the wing and of the Mach waves (Cf. Section I-K). However, at a subsonic edge the flow singularity is of the type encountered in subsonic flow. In the lifting case the perturbation velocity becomes infinite as the distance to edge to the power $(-1/2)$. In the symmetrical case the edge represents a logarithmic singularity (e.g., Cf. Sections II and III; and Ref. 11 for a general discussion of the principles).

Now consider a trailing edge in the lifting case. As the flow crosses the edge, w has to become zero. This is the principle of cancellation of lift behind the wing (Cf. condition 1.12b). If the edge is supersonic, the lift may be cancelled discontinously by appropriate Mach waves at the edge (Cf. Ref. 2 for a discussion of the implications of this possibility). At a subsonic trailing edge, however, a finite discontinuity is not possible. There are two mathematical possibilities: Either the magnitude of w decreases continuously on the wing to zero, or there is a discontinuous shift from ∞ to 0. The latter case would simply be a reversal of what happens at the leading edge. In accordance with condition (1.13) it will be assumed that the first possibility corresponds to the actual behavior of the fluid (Kutta condition). The following two implications of condition (1.13) will actually be of decisive use in solving special problems in this report:

On a flat lifting wing, $|w|$ decreases continuously to zero at a subsonic trailing edge and increases to infinity as the distance to the $(-1/2)$ power at a subsonic leading edge (Cf. Section III). (1.13')

Near a subsonic edge (leading or trailing) of a flat symmetrical wing, $|w|$ has a logarithmic singularity (Cf. Sections II-A and II-C). (1.13")

C. Invariance of Fundamental Equations Under Transformations

The concept of the invariance (or, more generally, of the transformation) of an equation under a mapping or transformation of variables is very important in physics and mathematics in general. In Sections I-C to I-G inclusive, the behavior of the linearized supersonic equations under certain mappings will be investigated. This study will furnish an understanding of the symmetry properties discussed in Section I-B, a method for treating yawed wings, and finally a basis for the theory of conical flow. Similar concepts will be used later in treating the Prandtl-Glauert transformation (Cf. Section I-M). On the whole there are two different types of results from an investigation of transformation properties: (a) certain symmetry properties of solutions and (b) methods for generating new solutions from a given one.

The invariance of an equation under a transformation may be explained in two different ways:

In the first interpretation x_i and x_i^* are considered to be different systems of coordinates for the same space. If the equation does not change its form when the coordinates x_i^* are introduced instead of x_i, then it is said to be invariant under the change of variables $x_i \rightarrow x_i^*$. Equation (1.1) would have to be

$$g_{x_{1*}x_{1*}} + g_{x_{2*}x_{2*}} - g_{x_{3*}x_{3*}} = 0$$

in order to remain invariant.

In the second interpretation $x_i \rightarrow x_i^*$ is considered to be a mapping from a space S to a space S* whereby the point $P = (x_1, x_2, x_3)$ is mapped on the point $P^* = (x_1^*, x_2^*, x_3^*)$. This correspondence between points in S and S* induces a correspondence between functions defined in the two spaces. To a function f in S there corresponds a function f* in S* defined by the rule that f* takes the same value at P* as f at P. Similarly, a relation or equation in S will correspond to a transformed relation or equation in S*. Examples of such correspondence are given in Sections I-F and I-M. If the transformed equation is invariant, i.e., the same as the original equation, one obtains a new solution f* of this equation for each given solution f. Thus the mapping is used for generating new solutions (this is the way the oblique transformation is going to be used). Even the case when the equation is not invariant is important. Here the mapping is used to generate solutions of the transformed equation (Cf. Section I-M for discussion of Prandtl-Glauert transformation).

Of these two interpretations of invariance, the second, though less familiar, is definitely preferable in order to put the discussion on a firm basis. The abstract presentation just presented will be illustrated by examples in Section I-D to G.

The following four transformations, which are of interest for the present report, will not be discussed:

1. Reflection in the plane of the wing.
2. Rotation around the x_3-axis.
3. Lorentz transformation in the plane of the wing (oblique transformation).
4. Uniform expansion around the origin.

D. Reflection in the Plane of the Wing

By reflection in the plane of the wing is meant the transformation

$$x_i \longrightarrow x_i^*, \text{ where } x_1^* = +x_1,\ x_2^* = -x_2,\ x_3^* = x_3 \qquad (1.14)$$

Equation (1.1) is obviously invariant under this transformation. Let $P^* = (x_1, -x_2, x_3)$ be the reflected point of $P = (x_1, x_2, x_3)$. Let $\Phi(x_1, x_2, x_3)$ be a solution of Equation (1.1) and define Φ^* as the function whose value at P^* is that of Φ at P. In other words, $\Phi^*(x_1, -x_2, x_3) = \Phi(x_1, x_2, x_3)$. In accordance with the general reasoning in Section I-C, Φ^* is also a solution of Equation (1.1) because of the invariance under reflection. Applying Equation (1.2), one sees that if the perturbation velocities at P are u, v, w, then at P^* the corresponding quantities $u^* = u, -v^* = v, w^* = w$.

Thus Equation (1.2) is invariant under transformation (1.14) only if v is replaced by $v^* = -v$ in the mapping. In this way a solution Φ has generated another solution Φ^*. If Φ is even in x_2, then the two solutions are identical; i.e. $\Phi(P^*) = \Phi^*(P^*)$, $u(P^*) = u^*(P^*)$, etc., and there results the case of a symmetrical flow discussed in Section I-B. The important thing is that, conversely, if the boundary conditions are consistent with symmetric flow (e.g., if the shape of a symmetric body is given), it follows from the invariance discussed that the solution also is a symmetric flow. Actually, this conclusion is justified only if the boundary conditions determine the solution uniquely. The logic of this proof of symmetry will be discussed in greater detail in connection with conical symmetry (Section I-G).

Similarly, since Equation (1.1) is homogeneous, $-\Phi^*$ is a solution if Φ^* is. Hence the solution $\Phi(x_1, x_2, x_3)$ generates the solution $-\Phi(x_1, -x_2, x_3)$. If the original Φ is odd in x_2, the two solutions are identical. This is the lifting case of Section I-B.

Of course, one may also study reflection in the X_2X_3-plane and the corresponding symmetry with respect to this reflection. This case is, however, much more obvious. The main result for symmetries with respect to this plane is that, if the body is symmetrical, then v and w are symmetrical and u antisymmetric. In particular $\partial w/\partial x_1 = 0$ in this plane.

E. Rotation Around the X_3-Axis

Obviously both Equations (1.1) and (1.2) are invariant under rotation around the X_3-axis. This is the basis for studying bodies with rotational symmetry. It is mentioned here only as a preparation for introducing the oblique transformation. The significant fact is that Equation (1.1) is not invariant under a rotation about any other axis.' Thus the x_3-axis is a preferred direction, whereas the x_1- and x_2-axes are equivalent. In incompressible flow, on the other hand, the equation corresponding to Equation (1.1) is obtained from Equation (1.1) by changing the minus sign ahead of the last term into a plus sign. The resulting Laplace equation is invariant under all rotations. One might then ask whether there are transformations which are analogous to general rotations but which leave Equation (1.1) invariant.

Now an ordinary rotation around a point Q is a transformation $P \rightarrow P^*$ which leaves the distance to Q unchanged: $|QP| = |QP^*|$. The answer to the question above will be found by introducing a new metric based on the concept of hyperbolic distance. A hyperbolic rotation around Q, also called a Lorentz transformation, will be a transformation which leaves the hyperbolic distance to Q unchanged. These concepts will next be defined and discussed.

F. Hyperbolic Distance, Lorentz Transformation, Oblique Transformation, and Yaw

The following definition of hyperbolic distance between the points $P = (x_1, x_2. x_3)$ and $\bar{P} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ will prove convenient for dealing with supersonic flow when the Mach number is equal to $\sqrt{2}$ and the free stream is parallel to the x_3-axis.

$$\text{Hyperbolic distance } P\bar{P} = \left[(x_3 - \bar{x}_3)^2 - (x_2 - \bar{x}_2)^2 - (x_1 - \bar{x}_1)^2\right]^{\frac{1}{2}} \qquad (1.15)$$

Note that the Mach cone from a point P may be defined as the locus of all points whose hyperbolic distance to P is equal to zero.

A hyperbolic rotation around Q, or a Lorentz transformation, is defined as a transformation $P \rightarrow P^*$ such that the hyperbolic distance QP = hyperbolic distance QP^*. It follows that Q is mapped on itself and that the Mach cone from Q is invariant in the sense that any point on this Mach cone is transformed into some other point on the same cone.

The following classical theorem will be the basis for the discussion in the remainder of Section I-F. It can easily be proved directly for the special cases which will actually be used.

Equation (1.1) is invariant under any Lorentz transformation. (1.16)

To obtain a more explicit description of these transformations, consider first the case where $x_3^* = x_3$. As is easily seen, invariance of hyperbolic distance is then equivalent to invariance of ordinary Euclidean distance. Thus such a transformation is simply an ordinary rotation around the x_3-axis, and theorem (1.16) reduces to the invariance discussed in Section I-E.

However, when $x_3 \neq x_3^*$, the hyperbolic rotation differs from the ordinary rotation. One will need only the case of a hyperbolic rotation around the origin which leaves the plane of the wing invariant. Such a special Lorentz transformation will be called an oblique transformation and its explicit expression is:

$$\left.\begin{aligned} x_1^* &= \frac{x_1 + ax_3}{\sqrt{1 - a^2}} \\ x_2^* &= x_2 \\ x_3^* &= \frac{ax_1 + x_3}{\sqrt{1 - a^2}} \end{aligned}\right\} \qquad -1 \leq a \leq 1 \tag{1.17}$$

The fact that this transformation leaves both the hyperbolic distance and Equation (1.1) invariant is easily checked. Theorem (1.16) depends, of course, on the fact that Equation (1.1) is a special case of the wave equation. The general equation for a disturbance (e.g. optic or acoustic,) spreading in (x,y,z)-space with velocity c is:

$$g_{xx} + g_{yy} + g_{zz} - \frac{1}{c^2} g_{tt} = 0 \tag{1.18}$$

This equation reduces to Equation (1.1) in the obvious way. In other words, if x_3 is thought of as denoting time, Equation (1.1) is the equation for a disturbance spreading in the X_1X_2-plane with unit velocity. This analogy was first noted and exploited by von Kármán (Cf. Ref. 12).

That Equation (1.18) is invariant under a four-dimensional Lorentz transformation is a well-known fact from mathematical physics and is fundamental in wave theory and theory of relativity. In aerodynamics it has been used by Kuesner and others (Cf. Ref. 13) for the study of nonstationary processes like fields due to moving sources. When Equation (1.18) is specialized to Equation (1.1), the general theory of Lorentz transformation yields the assertion (1.16), in particular the invariance of Equation (1.1) under Equation (1.17). Although this is an obvious consequence of von

Kármán's analogy, to the author's knowledge it has never been utilized in aerodynamics until recently when Jones (Cf. Ref. 14) used the oblique transformation (1.17) to treat a special case of a yawed wing. The idea of treating yawed wings by this method has proved quite fruitful; it will be discussed further and applied often in the present report.

Some formal properties of Equation (1.17) will now be developed. Since $x_2 = x_2{}^*$, the oblique transformation affects only the plane of the wing and may be represented by a two-dimensional matrix

$$\overline{M} = \frac{1}{\sqrt{1 - a^2}} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1{}^* \\ x_3{}^* \end{pmatrix} = \overline{M} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \qquad (1.17\ )$$

The Jacobian of this transformation is the determinant of $\overline{M}$ and is unity. Hence the oblique transformation is area-preserving. (1.19)

That this statement is of importance for evaluating lift of a yawed wing will be clear from the discussion later in Section I-F.

The inverse transformation of Equation (1.17) is given by

$$\left.\begin{aligned} x_1 &= \frac{x_1{}^* - a x_3{}^*}{\sqrt{1 - a^2}} \\ x_2 &= x_2{}^* \\ x_3 &= \frac{-a x_1{}^* + x_3{}^*}{\sqrt{1 - a^2}} \end{aligned}\right\} \qquad (1.20)$$

These equations mean that the inverse of $\overline{M}$ in Equation (1.17) is given by

$$\frac{1}{\sqrt{1-a^2}}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} \tag{1.20'}$$

Now let $\Phi(x_1, x_2, x_3)$ be a solution of Equation (1.1). Then in accordance with the general principle discussed in Section I-C, one may generate another solution from Φ. Let P^* be obtained from P by the transformation (1.17). Then put

$$\Phi^*(P) = \Phi(P^*) \tag{1.21a}$$

Using Equations (1.17), Equation (1.21a) may be written

$$\Phi^*(x_1,x_2,x_3) = \Phi\left(\frac{x_1 + ax_3}{\sqrt{1-a^2}}, x_2, \frac{ax_1 + x_3}{\sqrt{1-a^2}}\right) \tag{1.21b}$$

If u^* is the sidewash belonging to the generated solution[1]

$$u^*(P) = \left(\frac{\partial \Phi^*}{\partial x_1}\right)_P = \sum \left(\frac{\partial \Phi}{\partial x_i^*}\right)_{P^*} \cdot \left(\frac{\partial x_i^*}{\partial x_1}\right)_P = \frac{1}{\sqrt{1-a^2}}\left[u(P^*)+a\, w(P^*)\right]$$

Carrying out the same computation for the other velocity components yields

$$\left.\begin{aligned} u^*(P) &= \frac{1}{\sqrt{1-a^2}}\left[u(P^*) + a\, w(P^*)\right] \\ v^*(P) &= v(P^*) \\ v^*(P) &= \frac{1}{\sqrt{1-a^2}}\left[a\, u(P^*) + w(P^*)\right] \end{aligned}\right\} \tag{1.21c}$$

[1]Generally in this report u^*, v^*, and w^* will denote the harmonic conjugates of u, v, and w (Cf. Section I-J). The notation in Equations (1.21) and (1.22) will not be used later on.

Thus equation (1.2) is not invariant under Equation (1.17). As a matter of fact, Equation (1.2) is instead invariant under ordinary rotations. In general, $w^*(P) = w(P^*)$ contradicts Equation (1.21c). However, since w also obeys Equation (1.1), one may define a new solution w^* by

$$w^*(P) = w(P^*) \tag{1.22}$$

But if u^* and v^* are then computed from Equations (1.2) and (1.22), they will in general not obey Equation (1.21).

In the applications both schemes (1.21) and (1.22) will be used depending on the problem, the choice of scheme having to do with the boundary conditions at the edges. The discussion of this and some other important features of the oblique transformation will be presented in Section I-L.

G. Uniform Expansion. Conical Flow

By a uniform expansion around the origin is meant the transformation

$$x_i \rightarrow x_i^* = cx_i, \quad c > 0 \tag{1.23}$$

Evidently Equations (1.1) and (1.2b) are invariant under this transformation, but Equation (1.2a) only if Φ is replaced by $c\,\Phi$.

Invariance under reflection was related to certain symmetry properties as discussed in Section I-C. Similarly the invariance of the equations under Equation (1.23) will imply a certain conical symmetry of the flow wherever the boundary conditions themselves are invariant.

A flow field is said to be conical with respect to a point P (apex) if u, v, w, and thermodynamic quantities like pressure, density, etc. are constant along any half-line (ray) issuing from P. Unless otherwise specified, the apex will always be assumed to be the origin of the system of coordinates. Then the definition of conical flow above may be re-phrased: A flow field is conical if it is invariant under Equation (1.23). Finally one may say that a flow field is conical if u, v, etc. depend only on the ratios x_1/x_3, x_2/x_3.

A conical body with apex at the origin is formed when a half-infinite straight line with a fixed end point at the origin moves along some space curve. The simplest example is of course the circular cone, but an infinite flat triangle with apex at the origin is also a conical body.

Boundary conditions are said to be conical whenever they are invariant under Equation (1.23). Thus the same values of perturbation velocities or whatever is given must be prescribed for the point (x_1, x_2, x_3), and any point (cx_1, cx_2, cx_3) where $c > 0$. The simplest example of such boundary conditions occurs when the shape of a conical body is given. The other important example for the present report occurs when the pressure is prescribed in the plane of the wing in such a way that it is constant along each ray from the origin.

The basis for the theory of conical flow is the following assertion about conical symmetry;

> A flow field is conical if the boundary conditions are conical. (1.24)

This statement is actually valid for the full nonlinear Euler equations but fails when viscosity is introduced. Because of its fundamental importance the proof will be considered in some detail even though it may be claimed to be rather obvious. It may be proved in two different ways which are at least seemingly different: first, a physical argument based on dimensional analysis and, second, a more mathematical analysis dependent on the principles of invariance discussed in Section I-C.

The first argument runs as follows: In a nonviscous flow no combination of the quantities given at infinity (w_∞, P_∞ etc.) has the dimension length. Furthermore, a conical body or any conical boundary condition is described without specifying any length. Thus the problem contains no characteristic length, and the only way to form nondimensional combinations involving the coordinates is to divide these by each other; for example, u/w_∞ has to be a function of x_1/x_3, x_2/x_3, and some nondimensional combination of the values at infinity.

The second method of proving Equation (1.24) may seem more involved but will be given as a concrete example of the principles discussed in Section I-C. It should be emphasized that this method works in many cases where dimensional analysis fails (e.g., for the Prandtl-Glauert transformation). Consider a stationary supersonic flow around a body B (not necessarily conical) in a space S. Assume that the solution for this flow has been obtained. Now construct a new solution in another space S* by the following rule: To each point P in S let a point P* in S* correspond where P* is obtained from P by the transformation (1.23). The body B in S will then correspond to an expanded body B* in S*. Define a flow field in S* by letting the velocity at P* be the the same vector as that at P and by letting conditions at infinity be the same in both spaces.

The flow thus defined is the correct solution for the flow around B* in S* with the given conditions at infinity; therefore Equations (1.1) and (1.2b) and the boundary condition (1.4a) are fulfilled. The first part of this assertion follows from the invariance of the equations under the mapping rule (1.23). The second part follows from the fact that the normal at P* on B* has the same direction as the normal at P on B.

Now let the body be conical. Then B and B* are the same (or rather congruent) bodies. Hence assuming that the problem (equations and boundary conditions) has a unique solution, the flow field in S must be exactly the same as in S*. But from the way the flow field in S* was constructed, it follows that flow conditions at the point (cx_1, cx_2, cx_3) in S must be the same as those at (x_1, x_2, x_3). Since this statement is true for any positive c, the flow in S must be conical.

Thus assertion (1.24) is proved for the case when the shape of a conical body is prescribed. The proof for any type of conical boundary conditions is of course similar.

As is well known (Cf. Ref. 4), the above considerations apply also to Φ/x_3 (but not to Φ itself). However, very little use will be made of this fact in the present report.

H. Reductions of Equations for Linearized Supersonic Conical Flow. Tschaplygin Transformation

Although the reasoning in Section I-G was valid for the nonlinear Euler equations, the discussion will now be restricted to the linearized equations. As pointed out in Section I-G, in conical flow the number of independent space variables may be reduced from 3 to 2. This reduction will now be carried through explicitly for Equation (1.1).

Consider a solution $g(x_1, x_2, x_3)$ of Equation (1.1) where g is either u, v, or w (but not Φ). To say that g is invariant under Equation (1.23) is the same as saying that g is homogeneous of order zero in x_1, x_2, x_3. Introduce cylindrical coordinates (r, θ, x_3) by

$$\theta = \arc\tan \frac{x_2}{x_1} \tag{1.25a}$$

$$r = +\sqrt{x_1^2 + x_2^2} \tag{1.25b}$$

Coordinates θ and r are homogeneous of order 0 and 1, respectively, in x_1 and x_2. By repeated application of Euler's theorem for homogeneous functions

$$x_3 g_{x_3} = -\sum_{1,2} x_i g_{x_i} = -\sum_{1,2} (x_i g_\theta \cdot \theta_{x_i} + x_i g_r r_{x_i})$$

$$= -0 \cdot g_\theta - r \cdot g_r = -r \cdot g_r$$

Since $x_3 \cdot g_{x_3}$ is also homogeneous of order 0, by the same principle

$$x_3 \left[x_3 g_{x_3} \right]_{x_3} = -r(-r g_r)_r$$

Thus

$$x_3^2 g_{x_3 x_3} = x_3 \left[x_3 g_{x_3} \right]_{x_3} - x_3 g_{x_3} = r(r \cdot g_r)_r + r \cdot g_r \tag{1.26}$$

It is also known that the two-dimensional Laplacian may be expressed in polar coordinates as

$$g_{x_1 x_1} + g_{x_2 x_2} = \frac{1}{r}(r \cdot g_r)_r + \frac{1}{r^2} g_{\theta\theta} \tag{1.27}$$

Finally it is observed that a conical flow is completely known if it is known on the plane $x_3 = 1$. Hence a reduction of variables may be effected by introducing Equations (1.26) and (1.27) into Equation (1.1) and then putting $x_3 = 1$. The result is

$$\left(\frac{1}{r} - r\right)\left(r \cdot g_r\right)_r - r \cdot g_r + \frac{1}{r^2}\, g_{\theta\theta} = 0 \tag{1.28}$$

This is thus the equation for linearized conical flow at the station $x_3 = 1$. In this plane the Mach cone from the origin is represented by the unit circle (Mach circle). Parallel to the X_1-and X_2-axes in the plane $x_3 = 0$, the X_1- and X_2-axes may be introduced into the plane $x_3 = 1$ with the same coordinates x_1 and x_2. In this plane θ and r as introduced by Equation (1.25) become polar coordinates. The X_3-axis goes through the origin of the plane.

By applying the usual formal criteria (Cf. Ref. 15), one sees that Equation (1.28) is elliptic inside the unit circle and hyperbolic outside. An intuitive reason for this fact can be found by studying the domains of dependence of the original Equation (1.1). An interesting discussion of this question is given in Reference 4 .

Now the question arises: Since Equation (1.28) is elliptic inside the unit circle, does there exist a transformation of coordinates which reduces it to the simplest of all elliptic equations, namely, the Laplace equation? This question was first answered in the affirmative by Tschaplygin, who studied a similar equation in a different connection. A function A = Ts(a) for any real number between -1 and 1 may be defined by

$$Ts(a) = \frac{1 - \sqrt{1 - a^2}}{a} \tag{1.29}$$

This function will be called the Tschaplygin transformation of a. The change of coordinates introduced by Tschaplygin is then

$$\left.\begin{array}{l} \theta \quad \text{unchanged} \\ r \rightarrow R = Ts(r) \end{array}\right\} \tag{1.30}$$

It should be emphasized that there is a straightforward way of finding this transformation; namely, it is natural, first of all, to leave the angle θ unchanged since both Equation (1.28) and the Laplace equation are invariant under a rotation around the origin. Let $R = Ts(r)$ be unknown. Since $g_r = g_R \cdot R'$ and $g_{rr} = R'\left[g_{RR} \cdot R' + \frac{R''}{R'} g_R\right]$, where prime denotes differentiation with respect to r, Equation (1.28) becomes

$$R'(1 - r^2)(g_{RR}R' + \frac{R''}{R'} g_R) + (\frac{1}{r} - 2r)R' g_R + \frac{1}{r^2} g_{\theta\theta} = 0 \tag{1.28'}$$

In order to be the Laplace equation, formula (1.28) has to be identically equal to

$$g_{RR} + \frac{1}{R} g_R + \frac{1}{R^2} g_{\theta\theta} = 0 \tag{1.31}$$

Equations (1.28') and (1.31) are the same if the following two equations are true for any r:

$$(R')^2 r^2 (1 - r^2) = R^2 \tag{1.32a}$$

$$r^2(1 - r^2)R'' + r^2(\frac{1}{r} - 2r)R' = R \tag{1.32b}$$

The problem is solvable only if these equations are consistent. This is the case since Equation (1.32b) follows from Equation (1.32a) directly by differentiation. And Equation (1.32a) can be written

$$(\ln R)' = \frac{1}{r\sqrt{1-r^2}}$$

which immediately integrates to Equation (1.30) if the solution is normalized by requiring $Ts(1) = 1$.

Following Busemann, the Cartesian coordinates (x, y) are introduced corresponding to (R, θ) and also complex coordinates by

$$\zeta = x_1 + ix_2 = r \cdot e^{i\theta} \tag{1.33a}$$

$$\epsilon = x + iy = R \cdot e^{i\theta} \tag{1.33b}$$

The plane $X_3 = 1$ with its natural coordinates will be referred to as the ζ-plane; and the corresponding plane after the Tschaplygin transformation, as the ϵ-plane. The Tschaplygin transformation maps each point $\zeta = r \cdot e^{i\theta}$ in the ζ-plane on a point $\epsilon = R \cdot e^{i\theta}$ in the ϵ-plane where $R = Ts(r)$. Also ϵ will be denoted by $Ts(\zeta)$. This transformation leaves the unit circle and the origin invariant point by point and also transforms the rays from the origin into themselves, although not point by point; e.g., if ζ is real and positive, $Ts(\zeta)$ is real and positive but different from ζ except when $\zeta = 1$.

Busemann (Cf. Ref. 4) gives a geometrical interpretation of the Tschaplygin transformation and also points out that it may be written

$$\frac{1}{\zeta} = \frac{1}{2}\left(\bar{\epsilon} + \frac{1}{\epsilon}\right) \tag{1.34}$$

In the applications one is often interested especially in flow conditions in the plane of the wing. Then it is convenient to introduce (Cf. Fig. 1)

$$t = \frac{x_1}{x_3} \qquad \text{and} \qquad \tau = \text{arc tan } t \tag{1.35}$$

Either one of the coordinates t or τ specifies a conical ray in the plane of the wing uniquely; e.g., pressure distributions on such wings are functions of t (or τ) only. In the ζ -plane one has the simple relations on the X_1-axis:

$$t = r \cdot \cos\theta = x_1, \qquad r = |t| \tag{1.36a}$$

The Tschaplygin transformation of t is denoted by T

$$T = \frac{1 - \sqrt{1 - t^2}}{t} \tag{1.29'}$$

Then

$$T = R \cdot \cos\theta = x, \qquad R = |T| \tag{1.36b}$$

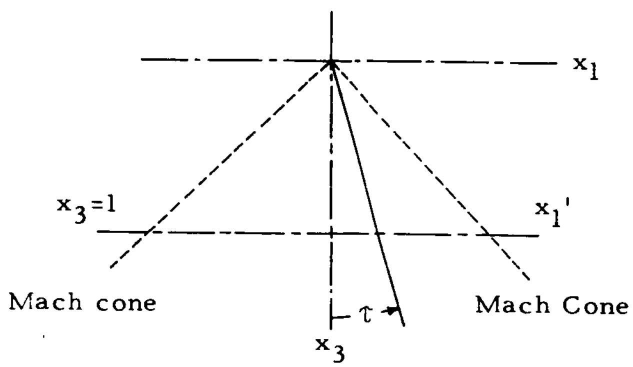


FIGURE 1. COORDINATES ON THE PLANE OF THE WING

I. Algebraic Properties of the Tschaplygin Transformation

Let A be Ts(a). In the application a will be r, t, or tangent of the angle of some side edge. Then Equation (1.29) implies the following useful relations between a and A:

$$\frac{2A}{1 + A^2} = a \qquad \text{(from Eq. 1.34)} \qquad (1.37a)$$

$$A + \frac{1}{A} = \frac{2}{a} \qquad \text{(from Eq. 1.37a)} \qquad (1.37b)$$

$$\frac{(1 + A)^2}{2A} = \frac{1}{a} + 1 \qquad (1.37c)$$

$$A^2 + \frac{1}{A^2} = \frac{2(2 - a^2)}{a^2} \qquad (1.37d)$$

$$\frac{1}{A} = \frac{1 + \sqrt{1 - a^2}}{a} \qquad (1.37e)$$

$$A^2 = \frac{1 - \sqrt{1 - a^2}}{1 + \sqrt{1 - a^2}} \qquad (1.37f)$$

$$\frac{1 - A^2}{1 + A^2} = \sqrt{1 - a^2} \qquad (1.37g)$$

$$\frac{A}{1 - A^2} = \frac{a}{2\sqrt{1 - a^2}} \qquad (1.37h)$$

$$\frac{1 \pm A}{\sqrt{A}} = \sqrt{\frac{2(1 \pm a)}{a}} \qquad (1.37i)$$

Also it may be seen from the binomial theorem that

$$A \doteq \tfrac{1}{2}a \text{ for } a \ll 1 \qquad (1.38)$$

J. Relations Between the Velocity Components in the ϵ -Plane

In Section I-H the following basic result taken from Busemann was proved:

> In conical flow, the velocity components u, v, w, satisfy Laplace equations when expressed as functions of the coordinates x and y defined by Equation (1.33b). (1.39)

This statement means that there must exist complex-valued functions U, V, W such that U, V, W are analytic functions of ϵ

$$
\begin{aligned}
U &= u + iu^* \\
V &= v + iv^* \\
W &= w + iw^*
\end{aligned}
\tag{1.40}
$$

where the real-valued functions u^*, v^*, w^* are the harmonic conjugates (determined only within a constant) of the potential functions u, v, and w. From now on, an asterisk on the symbol for a perturbation velocity will always denote the harmonic conjugate. The function U, V, W will be called the complex velocity components.

Since u, v, w are interconnected by Equation (1.2), there must exist some analogous relations for U, V, and W. These relations should follow from Equations (1.1) and (1.2) and the fact that the flow is conical. Busemann (Cf. Ref. 4) states the following result:

$$d(u + iv) = -\tfrac{1}{2}\left(\frac{d\overline{W}}{\overline{\epsilon}} + \epsilon\, dW\right) \tag{1.41}$$

This equation may be proved as follows: The right-hand side may be written as

$$
\begin{aligned}
-\tfrac{1}{2}\left[\frac{dw-idw^*}{Re^{-i\theta}} + Re^{i\theta}(dw+idw^*)\right] &= -\tfrac{1}{2}\, e^{i\theta}\left[\left(R + \frac{1}{R}\right)dw + i\left(R - \frac{1}{R}\right)dw^*\right] \\
&= -e^{i\theta}\left(\frac{dw}{r} - i\frac{R}{R'}\,\frac{1}{r^2}\, dw^*\right)
\end{aligned}
\tag{a}
$$

Use was here made of Equation (1.37b) and the formula obtained from Equation (1.37b) by taking the derivative with respect to r:

$$-\frac{2}{r^2} = \frac{R'}{R}\left(R - \frac{1}{R}\right) \tag{b}$$

Thus Equation (1.41) is equivalent to the two relations (c) and (d):

$$\frac{\partial(u + iv)}{\partial R} = -e^{i\theta}\left(\frac{1}{r}\frac{\partial w}{\partial R} + i\frac{1}{r^2}\frac{1}{R'}\frac{\partial w}{\partial \theta}\right) \tag{c}$$

or, multiplying by R',

$$\frac{\partial(u + iv)}{\partial r} = -e^{i\theta}\frac{1}{r}\left(\frac{\partial w}{\partial r} + i\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{c'}$$

and

$$\frac{\partial(u + iv)}{\partial \theta} = -e^{i\theta}\left[\frac{1}{r}\frac{\partial w}{\partial \theta} - i\frac{R}{R'}\frac{1}{r^2}\left(R\frac{\partial w}{\partial R}\right)\right] \tag{d}$$

or, using relation (b),

$$\frac{\partial(u + iv)}{\partial \theta} = -e^{i\theta}\left[\frac{1}{r}\frac{\partial w}{\partial \theta} - i(1 - r^2)\frac{\partial w}{\partial r}\right] \tag{d'}$$

In order to prove relation (c'), consider Equations (1.1) and (1.2) and homogeneity

$$\frac{\partial w}{\partial x_1} = \frac{\partial u}{\partial x_3} = -r\frac{\partial u}{\partial r}$$

$$\frac{\partial w}{\partial x_2} = \frac{\partial v}{\partial x_3} = -r\frac{\partial v}{\partial r}$$

Multiply the second equation by i and add

$$\text{grad } w = -r\frac{\partial(u + iv)}{\partial r}$$

which is relation (c')

For proof of relation (d'), consider Equations (1.1) and (1.2)

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} = \frac{\partial w}{\partial x_3} = -r\,\frac{\partial w}{\partial r}$$

$$\frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial x_1}$$

Making use of these equations to transform the operator $\frac{\partial}{\partial \theta}$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\,\frac{\partial u}{\partial x_1} + \cos\theta\,\frac{\partial u}{\partial x_2}$$

$$= \cos\theta\,\frac{\partial v}{\partial x_1} + \sin\theta\,\frac{\partial v}{\partial x_2} - \sin\theta\,\frac{\partial w}{\partial x_3}$$

$$= \frac{\partial v}{\partial r} - \sin\theta\,\frac{\partial w}{\partial x_3}$$

$$\frac{1}{r}\,\frac{\partial v}{\partial \theta} = -\sin\theta\,\frac{\partial v}{\partial x_1} + \cos\theta\,\frac{\partial v}{\partial x_2}$$

$$= -\left(\cos\theta\,\frac{\partial u}{\partial x_1} + \sin\theta\,\frac{\partial u}{\partial x_2}\right) + \cos\theta\,\frac{\partial w}{\partial x_3}$$

$$= -\frac{\partial u}{\partial r} + \cos\theta\,\frac{\partial w}{\partial x_3}$$

$$\frac{1}{r}\,\frac{\partial(u+iv)}{\partial \theta} = \frac{\partial(v-iu)}{\partial r} + \frac{\partial w}{\partial x_3}\,(i\cos\theta - \sin\theta)$$

$$= -i\,\frac{\partial(u+iv)}{\partial r} + ie^{i\theta}\,\frac{\partial w}{\partial x_3}$$

$$= -i\left[\frac{\partial(u+iv)}{\partial r} + e^{i\theta}\,r\,\frac{\partial w}{\partial r}\right]$$

Now use relation (c'):

$$\frac{\partial (u+iv)}{\partial \theta} = -ri\left[-e^{i\theta}\frac{1}{r}\left(\frac{\partial w}{\partial r}+i\frac{1}{r}\frac{\partial w}{\partial \theta}\right)+e^{i\theta}r\cdot\frac{\partial w}{\partial r}\right]$$

$$= -e^{i\theta}\left[-i\left(\frac{\partial w}{\partial r}+i\frac{1}{r}\frac{\partial w}{\partial \theta}\right)+i\,r^2\frac{\partial w}{\partial r}\right]$$

$$= -e^{i\theta}\left[\frac{1}{r}\frac{\partial w}{\partial \theta}-i\,(1-r^2)\frac{\partial w}{\partial r}\right]$$

which is relation (d').

It is sometimes convenient to transform Equation (1.41) to the form used in Reference 16.

Taking the complex conjugate of each side of Equation (1.41) yields

$$d(u-iv) = -\tfrac{1}{2}\left(\frac{dW}{\epsilon}+\bar{\epsilon}\,d\overline{W}\right)$$

Adding this to Equation (1.41) yields the relation

$$2du = -\tfrac{1}{2}\left[\left(\epsilon+\frac{1}{\epsilon}\right)dW+\overline{\left(\epsilon+\frac{1}{\epsilon}\right)dW}\right]$$

But

$$2du = dU + d\overline{U}$$

Hence

$$dU = -\tfrac{1}{2}\left(\epsilon+\frac{1}{\epsilon}\right)dW$$

Similarly

$$2idv = -\tfrac{1}{2}\left[\left(\epsilon-\frac{1}{\epsilon}\right)dW-\overline{\left(\epsilon-\frac{1}{\epsilon}\right)dW}\right]$$

and

$$2dv = dV + d\overline{V}$$

Hence it has been proved that

$$\frac{dU}{d\epsilon} = -\tfrac{1}{2}\left(\epsilon+\frac{1}{\epsilon}\right)\frac{dW}{d\epsilon} \qquad (1.42a)$$

and

$$\frac{dV}{d\epsilon} = + \frac{i}{2}\left(\epsilon - \frac{1}{\epsilon}\right)\frac{dW}{d\epsilon} \tag{1.42b}$$

Relations (1.42) will be called the compatibility relations. Three analytic functions U, V, and W give the three velocity components of a conical flow only if they satisfy these relations which take the place of the equations of continuity and irrotationality. When one of the three functions is given, relations (1.42) determine the other two only within a constant. The imaginary part of this constant is entirely arbitrary. The real part may be determined by the fact that $u = v = w = 0$ in the region of undisturbed flow and by the rules for the change of these quantities through a Mach wave or Mach cone (Cf. discussion following Eq. 1.44).

A very important consequence of relation (1.42b) was pointed out by Busemann (Cf. Ref. 4): Put $\left(\frac{dW}{d\epsilon}\right)_{\epsilon=0} = c$. If W is regular near the origin, then $dV/d\epsilon$ is there of the form

$$\frac{dV}{d\epsilon} = -\frac{ic}{2\epsilon} + \text{function regular at } \epsilon = 0$$

Hence

$$V = \frac{ic}{2}\ln\epsilon + \text{function regular at } \epsilon = 0 \tag{1.42c}$$

Similarly

$$U = -\frac{c}{2}\ln\epsilon + \text{function regular at } \epsilon = 0 \tag{1.42d}$$

Thus even if W is regular at the origin, U and V have logarithmic singularities there if $c \neq 0$. In particular, if c is real, i.e., $(dw^*)_{\epsilon=0} = 0$, then the upwash has a finite jump at the origin; v(P) increases suddenly by $(c\pi)/2$ if P passes through the origin on the X-axis in the negative direction.

This observation is of great importance in finding functions which satisfy the correct boundary conditions on the wing. It will be further discussed in Sections I-K and I-L, and examples of its application will be given in Sections II, V, and VI.

In the plane of the wing $\epsilon = T$. Hence, by Equation (1.37a)

$$\frac{1}{2}\left(\epsilon + \frac{1}{\epsilon}\right) = \frac{1 + T^2}{2T} = \frac{1}{t}$$

Thus in the plane of the wing

$$\frac{du}{dt} = -\frac{1}{t}\,\frac{dw}{dt}, \qquad t = \tan \tau = x_1 \tag{1.43a}$$

Similarly on the imaginary axis $\epsilon = iy$ and

$$\frac{i}{2}\left(\epsilon - \frac{1}{\epsilon}\right) = -\frac{1}{2}\,\frac{1 + y^2}{y} = -\frac{1}{x_2}$$

Hence in the X_2X_3-plane

$$\frac{dv}{dx_2} = -\frac{1}{x_2}\,\frac{dw}{dx_2} \tag{1.43b}$$

From these formulas it follows easily that if w is constant in a cónical region in the plane of the wing, u is also constant there. More generally, even when the flow is not conical, it follows directly from the equation of irrotationality that u is constant along a (finite or infinite) line segment parallel to the X_3-axis as long as $\partial w/\partial x_1 = 0$ on the segment. However, if w has a discontinuity on such a line, u might also change discontinuously. If the segment extends upstream into the undisturbed flow, the constant value is of course 0. Behind a trailing edge the constant might be different from zero, which for the lifting case means that there is a vortex line of constant strength. Examples of regions of constant w are, in the lifting case, the part of the plane wing not occupied by the wing and regions on the wings studied in Sections V and VI (Cf. Chap. 8-C of Reference 2).

K. Conical Wings: Classification. Boundary Conditions in Physical Space and in the ϵ -Plane

If a conical body is at the same time a wing as described in Section I-B, it will be called a conical wing. These are the only conical bodies studied in the present report. It will always be assumed that they are planar systems, i.e., that the boundary conditions may be applied to the plane of the wing. Of course one might also consider cases like wings with dihedral, but they will not be treated in this report. Such cases may easily be reduced to planar systems (Cf. Ref. 10).

Conical wings will be classified in various ways; for example, in Section I-B they were classified with respect to symmetry conditions of the flow or the nature of the quantities prescribed on the boundary. When the shape is given, it will generally be assumed that the wing is flat; i.e., v = constant. There are two such cases: (1) In Section II flat symmetrical wings will be treated. These have wedge-shaped profiles of constant half angle λ, the word profile meaning the intersection of the wing and a plane x_1 = constant. (2) In Section III the flat lifting wings of zero thickness are treated; the profiles are here straight lines of constant angle of attack α. After the flat wings, the wings with given pressure distribution will be considered. Since the lifting case is of most interest, in Section IV wings of constant-lift distribution (w = constant on wing) will be treated. In Section V wings of mixed type are treated. These are wings where the pressure is prescribed on some region of the wing and the shape on some other region. It will be assumed that the second region is flat and at zero angle of attack. In the first, w will be assumed to be constant. Only the lifting case is of interest. These wings give essentially an interference effect, the interference of a lifting element on a flat plate. Finally, Section VI treats another interference effect: that of a flat lifting wing on an adjacent flat plate.

Conical wings will also be classified according to planform. In all cases of interest the planform is bounded by two rays from the origin which may be considered to be the edges of an infinite triangle. These are

Planform I, in which both edges are outside the Mach cone.

Planform II, in which one edge is outside, one inside.

Planform III, in which both edges are inside.

For each of these cases subcases (a) and (b) are distinguished:

(a) Both edges are leading.

(b) One edge is leading, one trailing.

There is also an intermediate subcase (ab):

(ab) One edge is leading; the other is an unyawed side edge, i.e. parallel to the direction of the free stream.

For the boundary conditions at the subsonic edges, the statements in

Section I-B apply.

Consider now a supersonic leading edge with a sweepback angle ϕ. The sign of ϕ is defined arbitrarily so that ϕ is positive if the straight line passing through the edge passes through the first quadrant (x_1, $x_3 > 0$) of the plane of the wing. In Figure 2, ϕ is positive for the right-hand edge and negative for the other. In all cases treated in this report the region

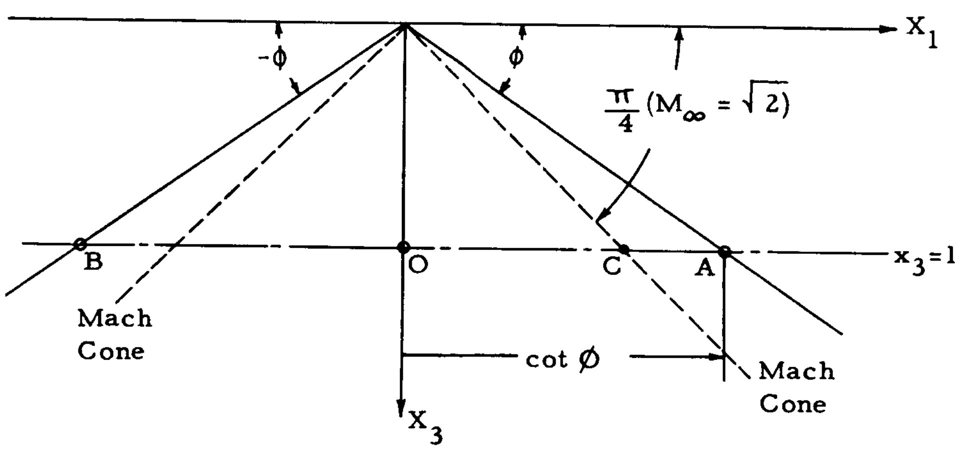


FIGURE 2. SUPERSONIC LEADING EDGES

of the wing outside the Mach cone from the origin will be flat since here w = constant is equivalent to v = constant. From the edge two plane Mach waves originate. In the region between these waves and the Mach cone from the origin, exact sweepback theory is valid; the velocity components are constant and given by

$$u = -w \tan \phi = \frac{-\alpha\, w_\infty \tan \phi}{\sqrt{1 - \tan^2 \phi}} \tag{1.44a}$$

$$v = -\alpha\, w_\infty \tag{1.44b}$$

$$w = \frac{\alpha\, w_\infty}{\sqrt{1 - \tan^2 \phi}} = \frac{-v}{\sqrt{1 - \tan^2 \phi}} \tag{1.44c}$$

It should be remembered that the free stream Mach number is assumed to be equal to $\sqrt{2}$.

The inclination of the leading edge Mach wave itself is given by sweepback theory and is such that its plane is tangent to the Mach cone from the leading edge.

Across the Mach cone a continuous change in the values of u, v, w is assumed. Hence on the Mach cone the perturbation velocities are given by Equations (1.44) for the part of the cone in contact with the above-mentioned regions between Mach cone and Mach waves. On the part of the Mach cone in contact with undisturbed fluid u, v, and w = 0. This condition holds true in particular for the entire Mach cone if the wing is completely inside the Mach cone. The boundary conditions above yield obvious boundary conditions in the ζ-plane. Consider Figure 3. The region between the upper surface, the Mach wave from the right-hand edge, and the Mach cone corresponds to the region ACD in Figure 3. The angle ψ between OD and OA is determined by the formula

$$\cos \psi = \tan \phi \tag{1.45}$$

This formula is easily derived from the fact that the trace of the Mach

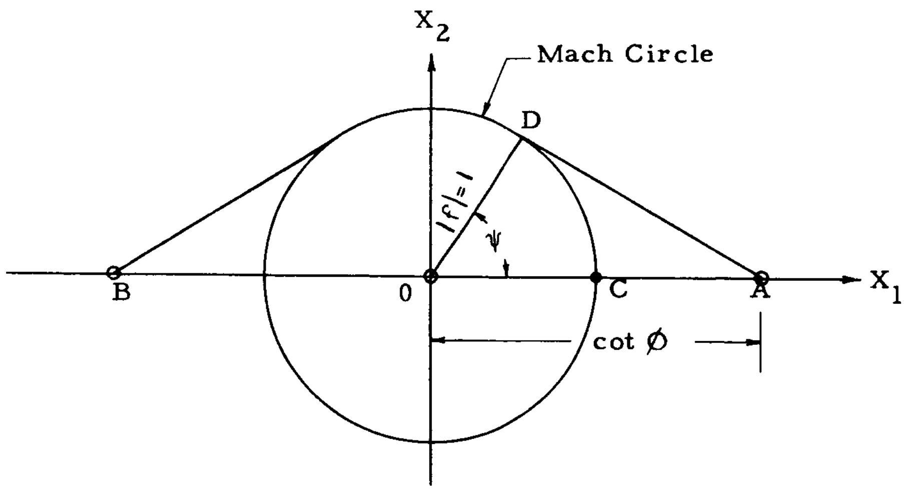


FIGURE 3. BOUNDARY CONDITIONS IN ζ -PLANE

wave in the ζ -plane (namely, AD) has to be tangent to the Mach circle. Thus if α is the angle of attack of the upper surface, u, v, and w are given by Equations (1.44) on the Mach circle for $0 \leq \theta \leq \psi$. For $\theta > \psi$, the perturbation velocities are zero till θ is large enough to bring the point $\zeta = e^{i\theta}$ under the influence of the left-hand wedge.

Since θ is unchanged by the Tschaplygin transformation, relation (1.45) is also valid in the ϵ -plane. Subsonic edges are represented as points inside the Mach circle on the X-axis, both in the ϵ -plane and in the ζ -plane. Condition (1.13') of Section I-B may be stated when applied, say, to w: At a trailing edge (which is represented by a point in the ϵ -plane) the analytic function W has a zero; at a leading edge $\epsilon = \epsilon_1$, this function may be written $W = 1/\sqrt{\epsilon - \epsilon_1}$ times a function analytic in at ϵ_1 .

In the ϵ -plane, however, one may make an essential reformulation of some boundary conditions. In particular the flatness condition α = constant may be given a very convenient formulation (Cf. Ref. 4):

$$\text{On a flat part of the wing } \partial w^*/\partial x = 0 \tag{1.46}$$

Consider a direct elementary proof of condition (1.46): Since α is constant, v is constant on the wing and $\partial v/\partial x_3 = 0$. Hence by irrotationality $\partial w/\partial x_2 = 0$ on the wing in the ζ -plane. Changing over to the ϵ -plane, one observes that on the X-axis, the x_2-direction coincides with the y-direction. Thus it follows from the symmetry of the Tschaplygin transformation that in the ϵ -plane the curves x_1 = constant cut the X-axis orthogonally. Hence $\partial w/\partial x_2 = 0$ implies $\partial w/\partial y = 0$. From the Cauchy-Riemann equations $\partial w^*/\partial x = 0$.

From Equation (1.46) Busemann concludes that w* = constant and that this constant then may be put equal to zero since w* is determined only within a constant anyway. However, if there are singularities in W, w* may take on different constant values on different segments. In particular, w* might take on different values on the top and bottom sides of a flat wing. This situation may be clarified by considering formula (1.42b), which shows that for real ϵ , i.e., in the plane of the wing, the downwash is connected with w* by

$$dv = -\left(\frac{\epsilon^2 - 1}{2\epsilon}\right) dw^* \tag{1.47}$$

Equation (1.47) furnishes a direct proof of Equation (1.46). Consider an edge when $\epsilon \neq 0$ and a path of integration consisting of a small circle with the edge as center. Integrate Equation (1.42b) along this path from one surface to the other and let the subscripts u and ℓ denote upper and lower surfaces. Then

$$V_u - V_\ell = +i \frac{\epsilon^2 - 1}{2\epsilon} (W_u - W_\ell) \tag{1.48a}$$

and taking the real part

$$v_u - v_\ell = -\left(\frac{\epsilon^2 - 1}{2\epsilon}\right)(w_u{}^* - w_\ell{}^*) \tag{1.48b}$$

Thus in the lifting case w* has the same value on the upper and lower surfaces. In the symmetrical case the difference $(w^*_u - w^*_\ell)$ is $2\lambda w_\infty$ divided by the real constant $(1 - \epsilon^2)/2\epsilon$, where λ is the half angle of the profile. Equation (1.48b) also shows that when $\epsilon = 0$, $w^*_u = w^*_\ell$ in order to keep the difference $v_u - v_\ell$ finite or zero.

Another possible singularity of v and w* is a discontinuity in slope on the surface of the wing. Section VI treats of wings where α takes on two constant values on the upper surface. Then $dv = dw^* = 0$ on the wing, but at the discontinuity w* has a jump which is obtained from the jump in v with the aid of Equation (1.47).

The discussion just given shows that in the following important case Busemann's statement is unconditionally true:

> On a flat wing of zero thickness and no discontinuity in shape, one may put $w^* = 0$ everywhere on the wing. (1.46')

One of Busemann's fundamental achievements was to point out the application of Equation (1.46′).

L. Oblique Transformation in ζ - and ϵ -Planes: Equivalence to a Homographic Transformation

In conical flow only the rays from the origin but not the individual points are of importance. Each ray (cx_1, cx_2, cx_3) where c runs through the positive numbers may be represented by the point

$$\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)$$

in the ζ -plane. The oblique transformation induces a certain transformation among the rays or among the points in the ζ -plane. Apply Equation (1.17) to a point $P = (x_1, x_2, 1)$ in the ζ-plane. P is transformed into the point

$$P* = \left(\frac{x_1 + a}{\sqrt{1 - a^2}}, x_2, \frac{1 + ax_1}{\sqrt{1 - a^2}}\right)$$

This point is on a ray which intersects the ζ-plane in the point

$$\left(\frac{x_1 + a}{1 + ax_1}, \frac{x_2\sqrt{1 - a^2}}{1 + ax_1}, 1\right)$$

Hence in the ζ-plane the oblique transformation (1.17) induces the transformation

$$x_1 \rightarrow x_1* = \frac{x_1 + a}{1 + ax_1} \tag{1.49a}$$

$$x_2 \rightarrow x_2* = \frac{x_2\sqrt{1 - a^2}}{1 + ax_1} \tag{1.49b}$$

In the applications of the oblique transformation one usually considers a transformation which takes a ray of angle β in the plane of the wing into the axis of the Mach cone. Such a transformation is obtained by putting a = - tan β in Equation (1.17). An arbitrary ray of angle τ in the same plane is then taken into a ray τ^* where

$$\tan \tau^* = \frac{\tan \tau - \tan \beta}{1 - \tan \beta \tan \tau} \tag{1.50}$$

This is the same formula as Equation (1.49a).

If the sweepback angle ϕ is measured as in Section I-K, formula (1.50) should be applied to $\left[\left(\frac{\pi}{2} - \phi\right)\right]$. Hence

$$\tan\left(\frac{\pi}{2} - \phi^*\right) = \frac{\tan\left(\frac{\pi}{2} - \phi\right) - \tan \beta}{1 - \tan \beta \tan\left(\frac{\pi}{2} - \phi\right)}$$

But because $\tan \left[(\pi/2) - \phi\right] = \cot \phi = (1/\tan \phi)$, this formula transforms into the same form as Equation (1.50)

$$\tan \phi^* = \frac{\tan \phi - \tan \beta}{1 - \tan \beta \tan \phi} \tag{1.50'}$$

where ϕ is the same as in Figure 2 and ϕ^* is the sweepback angle of the transformed edge. From Equation (1.50) there follows easily

$$\sqrt{1 - \tan^2 \tau^*} = \frac{\sqrt{1 - \tan^2 \beta}\sqrt{1 - \tan^2 \tau}}{1 - \tan \beta \tan \tau} \tag{1.51}$$

Now each mapping of the ζ-plane into itself induces a mapping in the ϵ-plane in the following way: If P is mapped on P_1 in the ζ-plane, then the induced mapping maps Ts(P) on Ts(P_1). Since Equation (1.1) is invariant under the Lorentz transformation, the Laplace Equation (1.31) must be invariant under the mapping which the oblique transformation induces in the ϵ-plane. This induced mapping is then conformal. It leaves unit circle and X-axis invariant and preserves direction on the X-axis. This invariance follows from the fact that both the oblique transformation and the Tschaplygin transformation possess these two properties. Furthermore, since in the ζ-plane the origin is mapped on a, in the ϵ-plane $Ts(0) = 0$ is mapped on

$$Ts(a) = A = \frac{1 - \sqrt{1 - a^2}}{a}$$

From the theory of analytic functions it then follows that the induced mapping must be the homographic transformation

$$\epsilon \rightarrow \eta = \frac{A + \epsilon}{1 + A\epsilon} \tag{1.52}$$

The determination of this function is actually a very simple application of two principles to be discussed in Sections II, III, and V, namely, extension of a function by reflection (in this case the unit circle) and determination of its analytical expression from its zeros, poles, and Riemann surface.

The following theorem has thus been proved:

A Lorentz transformation in the ζ-plane of the type (1.49), which takes the origin into the point $\zeta = a$, corresponds to a homographic transformation (1.52) in the ϵ-plane which takes the origin into

$$A = \frac{1 - \sqrt{1 - a^2}}{a} \tag{1.53}$$

Expressed in formulas, theorem (1.53) states that if

$$\zeta = x_1 + ix_2 \tag{a}$$

$$\zeta^* = \frac{1}{1 + ax_1}\left[(x_1 + a) + i\sqrt{1 - a^2}\; x_2\right] \tag{b}$$

$$\epsilon = Ts(\zeta) \tag{c}$$

$$A = Ts(a) \tag{d}$$

$$\epsilon^* = \frac{\epsilon + A}{1 + A\epsilon} \tag{e}$$

then

$$\epsilon^* = Ts(\zeta^*)$$

As a corollary, if Equation (1.52) is applied to a real ϵ $(\epsilon = T)$, one obtains the formula

$$\frac{T + A}{1 + TA} = \frac{1 + at - \sqrt{(1 - t^2)(1 - a^2)}}{t + a} \tag{1.54}$$

In the application it is sometimes convenient to use the algebraic identity (Cf. Eq. 1.37e):

$$\text{The right-hand side of Equation (1.54)} = \frac{t + a}{1 + at + \sqrt{(1 - t^2)(1 - a^2)}} \tag{1.54'}$$

For the benefit of the reader who distrusts an abstract proof, a computational proof of theorem (1.53) is supplied here. It is straightforward but longer than the proof just given.

Substituting Equation (a) in Equation (b) and multiplying the numerator and denominator by $\dfrac{4}{\zeta\bar{\zeta}}$

$$\zeta^* = \frac{2\left(\frac{1}{\zeta} + \frac{1}{\bar{\zeta}}\right) + \frac{4a}{\zeta\bar{\zeta}} + 2\sqrt{1 - a^2}\left(\frac{1}{\bar{\zeta}} - \frac{1}{\zeta}\right)}{\frac{4}{\zeta\bar{\zeta}} + 2a\left(\frac{1}{\zeta} + \frac{1}{\bar{\zeta}}\right)}$$

From Equations (c) and (d)

$$\frac{2}{\zeta} = \left(\frac{1}{\epsilon} + \bar{\epsilon}\right), \qquad a = \frac{2A}{A^2 + 1}$$

Hence

$$\zeta^* = \frac{\left(\frac{1}{\epsilon} + \frac{1}{\bar{\epsilon}} + \epsilon + \bar{\epsilon}\right) + \frac{2A}{1 + A^2}\left(\bar{\epsilon} + \frac{1}{\epsilon}\right)\left(\frac{1}{\bar{\epsilon}} + \epsilon\right) + \frac{1 - A^2}{1 + A^2}\left(\frac{1}{\bar{\epsilon}} + \epsilon - \frac{1}{\epsilon} - \bar{\epsilon}\right)}{\left(\bar{\epsilon} + \frac{1}{\epsilon}\right)\left(\epsilon + \frac{1}{\bar{\epsilon}}\right) + \frac{2A}{1 + A^2}\left(\frac{1}{\epsilon} + \frac{1}{\bar{\epsilon}} + \epsilon + \bar{\epsilon}\right)}$$

$$= \frac{(1 + A^2)\left[\epsilon + \bar{\epsilon} + \epsilon\bar{\epsilon}(\epsilon + \bar{\epsilon})\right] + 2A(1 + \epsilon\bar{\epsilon})^2 + (1 - A^2)\left[\epsilon - \bar{\epsilon} + \epsilon\bar{\epsilon}(\epsilon - \bar{\epsilon})\right]}{(1 + A^2)(1 + \epsilon\bar{\epsilon})^2 + 2A\left[\epsilon + \bar{\epsilon} + \epsilon\bar{\epsilon}(\epsilon + \bar{\epsilon})\right]}$$

$$= \frac{(1 + A^2)(\epsilon + \bar{\epsilon}) + 2A(1 + \epsilon\bar{\epsilon}) + (1 - A^2)(\epsilon - \bar{\epsilon})}{(1 + A^2)(1 + \epsilon\bar{\epsilon}) + 2A(\epsilon + \bar{\epsilon})}$$

$$= \frac{2\epsilon + 2A^2\bar{\epsilon} + 2A(1 + \epsilon\bar{\epsilon})}{1 + A^2 + \epsilon\bar{\epsilon} + A^2\epsilon\bar{\epsilon} + 2A(\epsilon + \bar{\epsilon})} = \frac{2(1 + \bar{\epsilon}A)(\epsilon + A)}{(1 + \epsilon A)(1 + \bar{\epsilon}A) + (\epsilon + A)(\bar{\epsilon} + A)}$$

$$= \frac{2\,\frac{\epsilon + A}{1 + \epsilon A}}{1 + \frac{\epsilon + A}{1 + \epsilon A}\cdot\frac{\bar{\epsilon} + A}{1 + \bar{\epsilon}A}} = \frac{2\epsilon^*}{1 + \epsilon^*\bar{\epsilon}^*}$$

Combining this formula with Equation (1.34) proves that $\epsilon^* = Ts(\zeta^*)$.

Let ϕ and ψ be defined as in Section I-K and related by Equation (1.45). A Lorentz transformation with parameter $a = -\tan\beta = -b$ transforms ϕ into an angle ϕ^* defined by Equation (1.50). The angle ψ is transformed into an angle Ψ^* which, by the previous reasoning, must satisfy the relations

$$\tan \phi^* = \cos \Psi^* \tag{1.55a}$$

$$e^{i\Psi^*} = \frac{e^{i\Psi} - B}{1 - Be^{i\Psi}} \tag{1.55b}$$

where

$$B = Ts(b) = \frac{1 - \sqrt{1 - b^2}}{b}$$

The following relations may be proved easily and will be of use in Section III-G:

$$\cos \frac{\Psi^*}{2} = \sqrt{\frac{1 - b}{2}} \sqrt{\frac{1 + \cos \Psi}{1 - b \cos \Psi}} \tag{1.56a}$$

$$\sin \frac{\Psi^*}{2} = \sqrt{\frac{1 + b}{2}} \sqrt{\frac{1 - \cos \Psi}{1 - b \cos \Psi}} \tag{1.56b}$$

In Section I-F methods for generating solutions by oblique transformation were discussed. Theorem (1.53) suggests that there should be analogous ways of generating solutions in the ϵ-plane by a homographic transformation.

Although the velocity potential itself does not appear as a function analytic in ϵ, there is still an analogue of the method of generating solutions by transformation of the velocity potential. Let U, V, and W be complex functions constituting a solution of some conical flow problem. In particular they have to satisfy the compatibility relations (1.42). Then the following three functions also satisfy relations (1.42) and constitute a solution for another conical problem:

$$U^a(\epsilon) = \frac{1 + A^2}{1 - A^2} U(\eta) + \frac{2A}{1 - A^2} W(\eta) \tag{1.57a}$$

$$V^a(\epsilon) = V(\eta) \tag{1.57b}$$

$$W^a(\epsilon) = \frac{2A}{1 - A^2} U(\eta) + \frac{1 + A^2}{1 - A^2} W(\eta) \tag{1.57c}$$

where

$$\eta = \frac{\epsilon + A}{1 + A\epsilon}$$

This conclusion follows from Equations (1.21c) and (1.37) and may also be checked by direct computation.

Using the method expressed in Equations (1.22), one may also put

$$W^a(\epsilon) = W(\eta) \tag{1.58}$$

It will be seen from examples to be given later (Cf., e.g., planform IIb in Section III-H) that a general formula like Equation (1.57) for constructing U^a and V^a is not possible. Hence these functions will have to be computed from W^a with the aid of Equation (1.42) in each separate case.

In the applications of Equation (1.58) the following question often arises: If W is the solution for a flat wing, is the same true for W^a? The hypothesis implies that

$$dv = dw^* = 0 \text{ on the wing} \tag{a}$$

From condition (a) and Equation (1.58) there follows:

$$dw^{a*} = dv = 0 \text{ on the corresponding wing} \tag{b}$$

Thus the second wing is flat (v = constant) unless V has a singularity there. If W is regular on the first wing, W^a is regular on the second wing. Then the only possible singularity of V is at $\epsilon = 0$. The nature of this singularity is given by Equation (1.42c). Thus the second wing is flat if it does not contain the origin.

M. Prandtl-Glauert Transformation Reduction to Mach Number $\sqrt{2}$

In the present report it is generally assumed that the free stream Mach number M is equal to $\sqrt{2}$. It will now be shown that within the linearized theory a wing problem at any other supersonic free stream Mach number may be reduced to an equivalent problem at $M = \sqrt{2}$.

Let the original problem be to find the perturbation velocity caused by a wing placed in a flow of free stream Mach number M. Denote by S^m the three-dimensional space in which this flow takes place. The problem will be reduced to the study of an equivalent wing placed in a free stream Mach number $= \sqrt{2}$ in a three-dimensional space S. (S and S^m are two copies of ordinary infinite three-dimensional Euclidean space.) A certain correspondence or mapping between the points in S and S^m will be set up which will in particular define the equivalent wing: It consists of the points in S on which the points of the wing in S^m are mapped. From the properties of this mapping there will follow a procedure for constructing the solution for the flow in S^m from that for the flow in S.

The mapping is the well-known Prandtl-Glauert transformation. A point P in S^m will be mapped on the corresponding point P* in S by the rule

$$P = (x_1, x_2, x_3) \longrightarrow P^* = (x_1, x_2, mx_3) \qquad (1.59)$$

where

$$m = \tan \mu = \frac{1}{\sqrt{M^2 - 1}}$$

This correspondence between points will induce a correspondence between functions. Two functions, f in S and f^m in S^m, are corresponding functions if

$$f^m(P) = f(P*) \tag{1.60}$$

Similarly, the two differential operators 0 and 0^m will correspond to each other if

> value of operator 0^m applied to f^m at P equals value of operator 0 applied to f at P*. (1.61)

Consider an example:

$$\text{If } 0 \text{ is } \frac{\partial}{\partial x_3}\text{ , then } 0^m = \frac{1}{m}\frac{\partial}{\partial x_3} \tag{1.62}$$

This correspondence may be proved as follows:

$$\frac{1}{m}\left(\frac{\partial f^m}{\partial x_3}\right)_P = \frac{1}{m}\lim_{\Delta x \to 0}\frac{f^m(x_1, x_2, x_3 + \Delta x) - f^m(x_1, x_2, x_3)}{\Delta x}$$

$$= \lim_{m\Delta x \to 0}\frac{f(x_1, x_2, mx_3 + m\Delta x) - f(x_1, x_2, mx_3)}{m\Delta x} = \left(\frac{\partial f}{\partial x_3}\right)_{P*}$$

The two wave operators Q and Q^m in the two spaces are defined by

$$Q = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

$$Q^m = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{1}{m^2}\frac{\partial^2}{\partial x_3^2}$$

By the method used immediately above it is proved that

Q corresponds to Q^m (1.63)

The definition (1.61) and the theorem (1.63) imply that:

If $Qf = 0$ at all points in S, then $Q^m f^m = 0$ at all points in S^m. (1.64)

In other words, if f is a solution of Equation (1.1), then f^m is a solution of Equation (1.1'). In this way solutions to Equation (1.1') may be generated from solution (1.1) by the rules (1.59) and (1.60). This procedure is very similar to the previously studied generation of function by the transformations mentioned in Section I-C. In the present case, however, Equation (1.1) is not invariant under the transformation considered, and for this reason the generated function f^m will not satisfy Equation (1.1) but the transformed Equation (1.1') instead.

In order to make use of relation (1.64), one has to investigate what boundary conditions the transformed function satisfies. Apply Equation (1.60) to the velocity potential ϕ, forming a potential ϕ^m for the flow in S^m. Form the velocity components u, u^m, v, etc. from these two potentials. One obtains the following relations (Cf. Eq. 1.62):

$$\left.\begin{aligned} \phi^m(P) &= \phi(P^*) \\ u^m(P) &= u(P^*) \\ v^m(P) &= v(P^*) \\ w^m(P) &= mw(P^*) \end{aligned}\right\} \qquad (1.65)$$

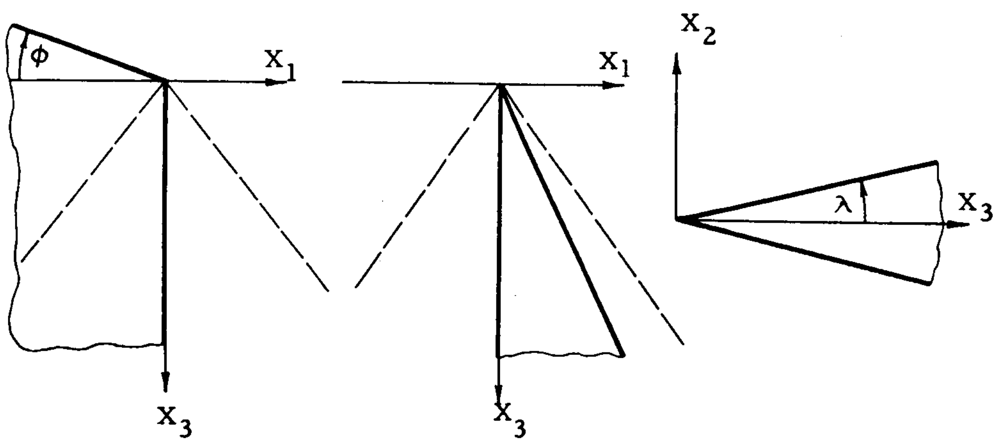
Now let there be given a wing in S^m of prescribed shape, satisfying Equation (1.6). If Equation (1.59) is applied literally to the wing, the equivalent wing has a different angle of attack because of the affine distortion. However, as explained in Section I-B, in the mathematical treatment the wing is replaced by its projection on the X_1X_3-space, which is given a certain distribution of α, i.e., of v. It is to this projected region that Equation (1.59) should be applied in forming the equivalent wing. It then follows from Equations (1.64) and (1.65) that Equation (1.65) is the correct rule for generating a solution for the original wing from that of the equivalent, provided the equivalent wing is prescribed to have the same angle of attack as the original wing at corresponding points.

An alternate scheme is used if the boundary conditions are given on w instead of on v. In this case let w have the same value at corresponding points of the two wings. Then one obtains the functions ϕ, μ, etc. for the equivalent wing, forms the functions ϕ^m, etc. as in Equation (1.65), and then multiplies them by the quantity m.

If one has mixed boundary conditions (Cf. Section I-B), the first of the above shcemes works if the only value prescribed for w is zero. Similarly, the second scheme works if the only value prescribed for v is zero.

II. FLAT SYMMETRICAL WINGS

This section discusses flat conical wings whose profile (i.e., intersection of wing and the plane x_1 = constant) is a wedge of half angle λ (Cf. Fig. 4). The wings are assumed to be symmetrical relative to the



(a) Planform IIab (b) Planform IIIab (c) Profile

FIGURE 4. BASIC CASES OF FLAT SYMMETRICAL WINGS

plane of the wing. The general characteristics of symmetrical wings were discussed in Section I-B. In the plane of the wing the boundary conditions are

$$\begin{aligned} &v = \lambda w_\infty \text{ on the upper surface of the wing} \\ &v = -\lambda w_\infty \text{ on the lower surface of the wing} \qquad (2.1) \\ &v = 0 \text{ off the wing} \end{aligned}$$

On the Mach cone the values of v are given as described in Section I-K.

As for boundary conditions at an edge, consider a subsonic edge, leading or trailing. In the ϵ-plane such an edge is represented by a point ϵ_1 $(-1 < \epsilon_1 < 1)$ and the boundary conditions in the neighborhood are as in Figure 5.

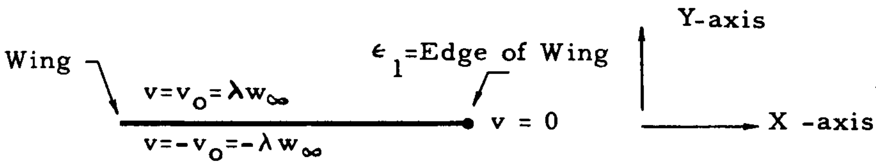


FIGURE 5. SUBSONIC EDGE OF SYMMETRICAL WING

It is seen that the given values of v are v_o/π times the polar angle θ with apex at ϵ_1 ; namely, if $\theta = 0, +\pi$, or $-\pi$, then $v = 0$, v_o, and $-v_o$, respectively. The analytical expression for the fact that v is proportional to θ is that it is the imaginary part of a logarithmic singularity at ϵ_1. More precisely

$$V = -i\,\frac{v_o}{\pi}\,\ell n(\epsilon - \epsilon_1) + f(\epsilon) \tag{2.2}$$

where $f(\epsilon)$ is analytic at ϵ_1 and purely imaginary along the X-axis. Thus $f(\epsilon)$ does not contribute anything to $v = \mathrm{Re}(V)$ there.

From Equation (1.42b) it follows that

$$\frac{dW}{d\epsilon} = -\frac{v_o}{\pi}\,\frac{2}{\epsilon^2 - 1}\cdot\frac{\epsilon}{\epsilon - \epsilon_1} + \frac{2\epsilon}{i(\epsilon^2 - 1)}\,f'(\epsilon) \tag{2.3}$$

Hence for $\epsilon_1 \neq 0$, W also has a logarithmic singularity. However, w is the real part of the logarithmic singularity and hence becomes logarithmically infinite at $\epsilon = \epsilon_1$, whereas w* behaves qualitatively like v. In the exceptional case where the side edge is along the axis of the Mach cone, $\epsilon_1 = 0$ and V still has the same type of singularity but W is regular.

Although the function (2.2) fits the boundary conditions in Figure 5, there are other possible solutions. These will have to be ruled out by special considerations. This subject will be discussed in Section II-B.

As pointed out in Section I-B, the fact that there are sufficient boundary conditions for one perturbation velocity makes the problem of finding solutions for symmetrical planar wings very easy. Another circumstance which simplifies the problem is that there is only pressure interference but no downwash interference between two such wings. Hence the boundary conditions are not affected by the interference. Thus in the present case it is sufficient to obtain the solution for planforms IIab and IIIab (Cf. Fig. 4). The other solutions for planforms I, II, and III may then be obtained by superposition. In particular the solution discussed in Reference 17 may be derived in this way.

A. Planform IIab

The boundary conditions are shown in Figure 6. As has just been explained, the singularity at the edge may be expressed by the function

$$\frac{-i\, v_o}{\pi} \ln \epsilon \qquad \text{(a)}$$

This function gives the correct values of v on the X-axis provided the

ϵ -plane is cut along the negative real X-axis. The two discontinuities on the Mach Circle may also be obtained as logarithmic singularities. If ϵ

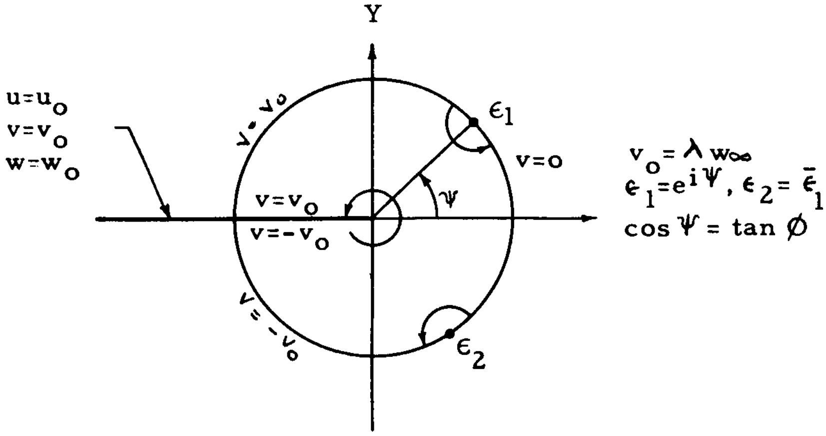


FIGURE 6. BOUNDARY CONDITION IN ϵ -PLANE

varies in the positive direction on a small half circle inside the Mach circle (Cf. Fig. 6), then v decreases by v_o. This type of discontinuity suggests that V is of the form

$$\frac{iv_o}{\pi} \ln(\epsilon - \epsilon_1) + \text{function analytic at } \epsilon_1 \qquad \text{(b)}$$

Combining functions (a), (b), and the corresponding function for the discontinuity at ϵ_2 gives

$$V = \frac{iv_o}{\pi}\left[\ln(\epsilon - \epsilon_1)(\epsilon - \epsilon_2) - \ln\epsilon\right] \qquad \text{(c)}$$

Actually the function defined by (c) satisfies all the boundary conditions. Since

$$\text{Re}\left[-i\ln(\epsilon - \epsilon_1)\right] = \arg(\epsilon - \epsilon_1)$$

and on the real axis arg $(\epsilon - \epsilon_1) = -\arg(\epsilon - \epsilon_2)$, the first term gives no contribution to v on the real axis, and the second term gives the correct boundary conditions. That conditions on the Mach circle are also satisfied will follow later on (Cf. proof of Eq. 2.8) when v is evaluated in physical coordinates (r, θ) from Equation (c).

But first U and W will be evaluated from Equations (c) and (1.42). From Equation (c)

$$\frac{dV}{d\epsilon} = \frac{iv_o}{\pi}\left[-\frac{1}{\epsilon} + \frac{1}{\epsilon - \epsilon_1} + \frac{1}{\epsilon - \epsilon_2}\right] \tag{d}$$

From relation (d) and Equation (1.42b)

$$\begin{aligned}\frac{dW}{d\epsilon} &= -\frac{2v_o}{\pi}\frac{\epsilon}{\epsilon^2 - 1}\left[\frac{1}{\epsilon} - \frac{1}{\epsilon - \epsilon_1} - \frac{1}{\epsilon - \epsilon_2}\right] \\ &= \frac{2v_o}{\pi}\left[\frac{1}{(\epsilon - \epsilon_1)(\epsilon_1 - \epsilon_2)} + \frac{1}{(\epsilon - \epsilon_2)(\epsilon_2 - \epsilon_1)}\right]\end{aligned} \tag{e}$$

Since $\epsilon_1 = e^{i\psi}$, $\epsilon_2 = e^{-i\psi}$

$$\epsilon_1 - \epsilon_2 = 2i\sin\psi \tag{f}$$

Combining relations (e) and (f),

$$W = \frac{iv_o}{\pi\sin\psi}\,\ell n\frac{\epsilon - \epsilon_2}{\epsilon - \epsilon_1} + c \tag{g}$$

To evaluate the constant of integration c, one observes that, according to (1.44) and (1.45),

at $\epsilon = -1$, $\quad \mathrm{Re}(W) = w = w_o = \dfrac{-v_o}{\sin\psi}$

But as may be seen by elementary geometry

$$\mathrm{Re}\left[-i\,\ell n\,\frac{-1-\epsilon_1}{-1-\epsilon_2}\right] = \Psi$$

Hence

$$\frac{\pi W}{w_o} = -i\,\ell n\,\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1} + \Psi + \pi \qquad \text{(h)}$$

Similarly U may be evaluated

$$\frac{dU}{d\epsilon} = -\frac{1}{2}\left(\epsilon+\frac{1}{\epsilon}\right)\frac{dW}{d\epsilon} = \frac{-i\,v_o}{2\pi\sin\Psi}\left(\epsilon+\frac{1}{\epsilon}\right)\left[\frac{-1}{\epsilon-\epsilon_1}+\frac{1}{\epsilon-\epsilon_2}\right] \qquad \text{(i)}$$

After some algebraic manipulations this formula becomes

$$\frac{dU}{d\epsilon} = \frac{-i\,v_o}{\pi}\left[-\frac{\cos\Psi}{\sin\Psi}\cdot\frac{1}{\epsilon-\epsilon_1}+\frac{\cos\Psi}{\sin\Psi}\cdot\frac{1}{\epsilon-\epsilon_2}-\frac{i}{\epsilon}\right] \qquad \text{(j)}$$

Integrating formula (j), using Equations (1.44) and (1.45), and determining the constant as for relation (g)

$$U = \frac{u_o}{\pi}\left[-i\,\ell n\,\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1}+\Psi+\pi\right] - \frac{v_o}{\pi}\,\ell n\,\epsilon \qquad \text{(k)}$$

Next u, v, and w will be evaluated as functions of the physical coordinates r and θ. Consider an arbitrary point $re^{i\theta}$ in the ζ-plane and the corresponding point $Re^{i\theta}$ in the ϵ-plane.

Using the notation in Figure 7 .

$$\left.\begin{aligned} \mathrm{Re}(-i\,\ell n(\epsilon_1-\epsilon)) &= \arg(\epsilon_1-\epsilon) = \phi_1 \\ \mathrm{Re}(-i\,\ell n(\epsilon_2-\epsilon)) &= \arg(\epsilon_2-\epsilon) = \phi_2 \end{aligned}\right\} \qquad (\ell)$$

These angles are connected with R and θ by

$$\left.\begin{aligned} \tan\phi_1 &= \frac{\sin\psi - R\sin\theta}{\cos\psi - R\cos\theta} \\ \tan\phi_2 &= \frac{-\sin\psi - R\sin\theta}{\cos\psi - R\cos\theta} \end{aligned}\right\} \qquad (m)$$

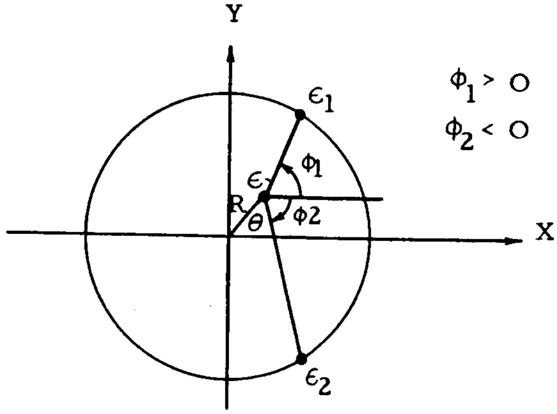


FIGURE 7. NOTATION FOR EVALUATION OF u, v, AND w IN ϵ-PLANE

From relations (m)

$$\tan(\phi_1 + \phi_2) = \frac{2\,R\sin\theta\,(R\cos\theta - \cos\psi)}{1 + R^2 - 2R^2\sin^2\theta - 2R\cos\theta\cos\psi} \qquad (n)$$

$$\tan(\phi_1 + \phi_2) = \frac{\sin\theta\left[(1 - \sqrt{1 - r^2})\cos\theta - r\cos\psi\right]}{1 - (1 - \sqrt{1 - r^2})\sin^2\theta - r\cos\theta\cos\psi} \qquad (n')$$

In a similar way one may derive

$$\cos(\phi_2 - \phi_1 + \psi + \pi) = \frac{r\cos\theta - \cos\psi}{\sqrt{(1 - r\cos\theta\cos\psi)^2 - r^2\sin^2\theta\sin^2\psi}}$$

The results derived are summarized in Equations (2.4) to (2.7):

For a wing of planform IIab, sweepback ϕ, and symmetrical wedge profile of half angle λ, the perturbation velocities are as follows. On the wing outside the Mach cone

$$u = u_o = -w_o \tan\phi = \frac{\lambda w_\infty \tan\phi}{\sqrt{1-\tan^2\phi}} \tag{2.4a}$$

$$\begin{aligned} v &= v_o = \lambda w_\infty \quad \text{on the upper surface} \\ v &= -v_o = -\lambda w_\infty \quad \text{on the lower surface} \end{aligned} \tag{2.4b}$$

$$w = w_o = \frac{-v_o}{\sqrt{1-\tan^2\phi}} = \frac{-\lambda w_\infty}{\sqrt{1-\tan^2\phi}} \tag{2.4c}$$

Inside the Mach cone the complex expressions for the velocity are

$$U = \frac{u_o}{\pi}\left[-i\,\ell n\,\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1} + \psi + \pi\right] - \frac{v_o}{\pi}\,\ell n\,\epsilon \tag{2.5a}$$

$$V = \frac{iv_o}{\pi}\left[\ell n(\epsilon-\epsilon_1)(\epsilon-\epsilon_2) - \ell n\,\epsilon\right] \tag{2.5b}$$

$$W = \frac{w_o}{\pi}\left[-i\ell n\,\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1} + \psi + \pi\right] \tag{2.5c}$$

Here $\cos\psi = \tan\phi$, $\epsilon_1 = e^{i\psi}$, $\epsilon_2 = \bar{\epsilon}_1$.

Putting $\arg(\epsilon_1 - \epsilon) = \phi_1$, $\arg(\epsilon_2 - \epsilon) = \phi_2$ (Cf. Fig. 7), and $\epsilon = R \cdot e^{i\theta}$, the real parts of the functions above may be written

$$u = \frac{u_o}{\pi}\left[\phi_2 - \phi_1 + \psi + \pi\right] - \frac{v_o}{\pi}\ell n\,R \tag{2.6a}$$

$$v = \frac{v_o}{\pi}\left[\theta - (\phi_1 + \phi_2)\right] \tag{2.6b}$$

$$w = \frac{w_o}{\pi}\left[\phi_2 - \phi_1 + \psi + \pi\right] \tag{2.6c}$$

With the aid of relations (n') and (o) the values of the brackets in Equations (2.6) may be given as a function of the physical coordinates r and θ

$$\cos(\phi_2 - \phi_1 + \psi + \pi) = \frac{r\cos\theta - \cos\psi}{\sqrt{(1 - r\cos\theta\cos\psi)^2 - r^2\sin^2\theta\sin^2\psi}} \tag{2.7a}$$

$$\tan(\phi_1 + \phi_2) = \frac{\sin\theta\left[(1 - \sqrt{1 - r^2})\cos\theta - r\cos\psi\right]}{1 - (1 - \sqrt{1 - r^2})\sin^2\theta - r\cos\theta\cos\psi} \tag{2.7b}$$

For points in the plane of the wing Equations (2.6) and (2.7) may be simplified considerably. Use is thereby made of the fact that θ is either $0, \pi$, or $-\pi$ and hence $\sin\theta = 0$ and $r\cos\theta = t$ (Cf. Eq. 1.36).

In the plane of the wing

$$u = \frac{u_o}{\pi}\operatorname{arc\,cos}\frac{t - \tan\phi}{1 - t\cdot\tan\phi} - \frac{v_o}{\pi}\ln\frac{1 - \sqrt{1 - t^2}}{|t|} \tag{2.8a}$$

$$v = \frac{v_o}{\pi}\theta \tag{2.8b}$$

$$w = \frac{w_o}{\pi}\operatorname{arc\,cos}\frac{t - \tan\phi}{1 - t\cdot\tan\phi} \tag{2.8c}$$

Finally on the Mach cone Equation (2.7b) reduces to $\tan\theta = \tan(\phi_1 + \phi_2)$. Hence $\theta - (\phi_1 + \phi_2) = n\pi$ where n is an integer. At the point $\zeta = 1$, n must be zero by Equation (2.8b). Hence by analytical continuation it remains zero on the Mach circle up to singularities ϵ_1 and ϵ_2. Here the jump is found by the method described at the beginning of Section II. In this way it is seen that Equation (2.5b) actually satisfies the boundary conditions on the Mach circle. This fact could also have been seen geometrically from Equation (2.6b).

B. Planform IIIab. Principle of Reflection

The boundary conditions as illustrated in Figure 8 are very similar to

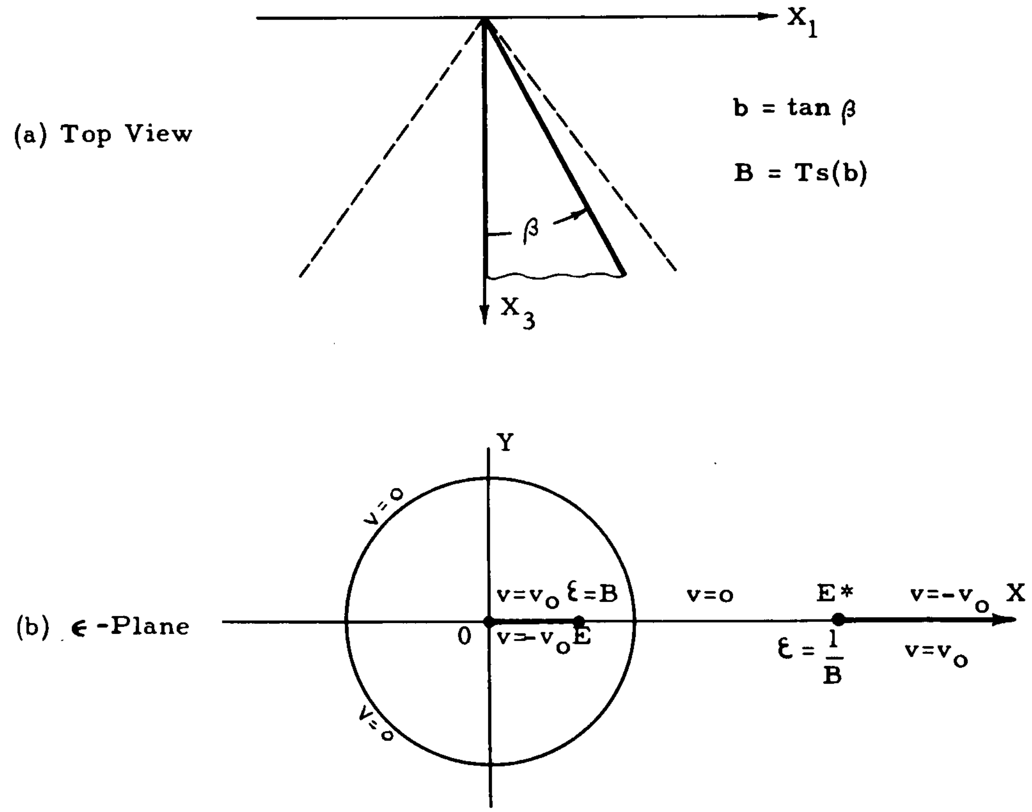


FIGURE 8. SYMMETRICAL WING: PLANFORM IIIab

those in Section II. The main difference is that the logarithmic singularities are all on the real axis of the ϵ-plane. Since the wing is entirely within the Mach cone, there are no singularities on the Mach circle; here $u = v = w = 0$. The function V has two logarithmic singularities inside the Mach circle, at $\epsilon = 0$ and $\epsilon = B$, which points correspond to the two edges. However, these two singularities are not sufficient to determine V. Some information about the behavior of V in the rest of the ϵ-plane will be necessary and can be secured with the aid of the principle of reflection.

A proof and exact formulation of this principle may be found in textbooks on analytic functions, especially those with the geometric (Riemann) approach (Cf. p. 218 of Ref. 18 or p. 372 of Ref. 19). The principle may be conveniently formulated in geometrical terms. One may consider the solution $V(\epsilon)$ as a mapping of the ϵ-plane into the V-plane; to each point $x + iy$ in the ϵ-plane there corresponds a point $V(\epsilon) = v + iv^*$ in the V-plane. In this way a straight line or a curve in the ϵ-plane will correspond to some curve or straight line in the V-plane.

The principle of reflection (Cf. Fig. 9) indicates that if a line L_1 is mapped analytically on a line L_2 and the mapping is defined on one side of L_1, then it may be extended to the other side of L_1 in such a way that points which are symmetrically located (images) with respect to L_1 are mapped on points which are symmetrically located with respect to L_2.

In this statement any one of the straight lines may be replaced by a circle if by image of a point with respect to a circle is meant the reciprocal point. Two points are reciprocal with respect to a circle if they

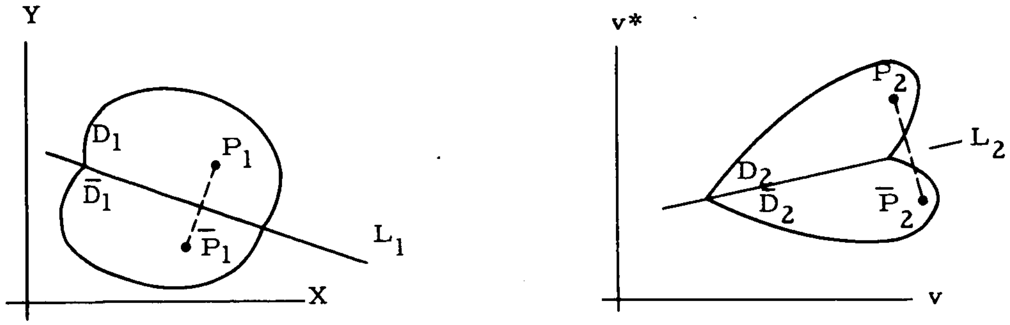


FIGURE 9. PRINCIPLE OF REFLECTION

are on the same ray from the center of the circle and the product of their distances to this center is equal to the square of the radius.

The principle of reflection will be of fundamental importance for many solutions given in this report. It will now be applied to the present example. On the unit circle in the ϵ -plane, v = 0; in other words, V is purely imaginary. Expressed in geometrical terms this fact means that the unit circle in the ϵ -plane is mapped on the imaginary axis in the V- plane. That ϵ_1 is the image of ϵ_2 in the unit circle means algebraically that $\epsilon_1 = 1/\bar{\epsilon}_2$. Similarly, if V_1 is the image of V_2 with respect to the imaginary axis, then $V_1 = -\bar{V}_2$. Hence V must satisfy the relation

$$V(1/\bar{\epsilon}) = -\overline{V(\epsilon)} \tag{2.9}$$

More precisely it is known that V is analytic inside the unit circle except for the slit OE (Cf. Fig. 8). Since v = 0 on the circumference of the unit circle, V may be extended to the outside of the unit circle by the rule (2.9), and it will be analytic there except at the slit which is the image of OE. The reciprocal of the point E: $\epsilon = B$ is the point E*: $\epsilon = 1/B$. The reciprocal of the origin is the point at infinity. Hence the image of OE is the slit extending along the real axis from $1/B$ to infinity. From Equation (2.9) there follows that $v = -v_o$ on top of the slit and $v = +v_o$ on the lower side. Thus there is the same type of logarithmic singularity as at the edge E. One may think of E* as the edge of the reflected wing. When ϵ varies counterclockwise on a small circle around E*, V increases by $2v_o$. This fact determines the strength of the singularity. Hence all singularities of V are known, and the solution may be written down immediately as for planform IIab:

$$V = \frac{-i\, v_o}{\pi}\left[\ln(\epsilon - B)(B\epsilon - 1) - \ln\epsilon\right] \tag{2.10}$$

$$v_o = \lambda w_\infty$$

where λ = half angle of profile. The functions U and W are then easily computed from Equations (2.10) and (1.42):

$$U = \frac{v_o}{\pi}\left[\ln\epsilon - \left(\frac{1+B^2}{1-B^2}\right)\ln\frac{\epsilon - B}{B\epsilon - 1}\right] \tag{2.11a}$$

$$W = \frac{v_o}{\pi}\left(\frac{2B}{1-B^2}\right)\ln\frac{\epsilon - B}{B\epsilon - 1} \tag{2.11b}$$

That the constant of integration has been chosen correctly may be seen from the fact that if $\epsilon = 1$, Re(U) = Re(W) = 0 which is the correct boundary v' ue. It is interesting to note as an additional check that

$\eta = (\epsilon - B)/(B\epsilon - 1)$ is the homographic transformation previously encountered in a different context (Cf. Section I L). This transformation maps the unit circle of the ϵ-plane into the unit circle of the η-plane. Hence when ϵ is on the unit circle, the arguments of the logarithmic functions in Equation (2.11) are applied to numbers of absolute value = unity, and u and w vanish on the entire unit circle.

For an arbitrary point $re^{i\theta}$ in the ζ-plane, w has the value

$$w = \frac{v_o}{\pi} \frac{2B}{1 - B^2} \ln \left| \frac{Re^{i\theta} - B}{BRe^{i\theta} - 1} \right| \tag{2.12a}$$

where (Cf. Eqs. 1.29 and 1.37h)

$$v_o = \lambda w_\infty , \quad R = Ts(r), \quad B = Ts(b), \text{ and } \quad \frac{2B}{1 - B^2} = \frac{b}{\sqrt{1 - b^2}}$$

In the plane of the wing $\epsilon = Re^{i\theta} = T = Ts(t)$. Hence

$$w = \frac{v_o}{\pi} \frac{2B}{1 - B^2} \ln \left| \frac{T - B}{BT - 1} \right| = \frac{v_o}{\pi} \frac{b}{\sqrt{1 - b^2}} \ln \left| \frac{1 - bt - \sqrt{(1-b^2)(1-t^2)}}{t - b} \right| \tag{2.13a}$$

Here use has been made of Equation (1.54). Similarly

$$u = \frac{v_o}{\pi} \left(\ln R - \frac{1 + B^2}{1 - B^2} \ln \left| \frac{Re^{i\theta} - B}{BRe^{i\theta} - 1} \right| \right) \tag{2.12b}$$

and in the plane of the wing

$$u = \frac{v_o}{\pi} \left(\ln |T| - \frac{1}{\sqrt{1 - b^2}} \ln \left| \frac{1 - bt - \sqrt{(1 - b^2)(1 - t^2)}}{t - b} \right| \right) \tag{2.13b}$$

C. Solutions with Higher-Order Singularities: Uniqueness

Some comments will now be made about the uniqueness of the solutions obtained in Sections II-A and B.

Consider, e.g., planform IIab. Denote by V_1 the solution obtained in Section II-B and by V_2 another solution with the same boundary condition. The difference $V_3 = V_1 - V_2$ then satisfies the following boundary condition:

$$v_3 = \mathrm{Re}(V_3) = 0 \text{ on real axis and on unit circle} \tag{2.14}$$

If the function V_3 is to be regular in the interior of the unit circle, condition (2.14) implies that it is identically zero (or equals an imaginary constant). In this case $V_1 = V_2$ except for irrelevant imaginary constants.

Nonvanishing solutions of condition (2.14) may of course be had by assuming various unnatural singularities of V_3. However, assume that the only singularity is at the origin $\epsilon = 0$ and that except for this point V_3 is regular in the unit circle.

Then by the principle of reflection V_3 is regular on the whole Riemann sphere (ϵ-plane and point at infinity) except for the singularity at $\epsilon = 0$ and the reflected singularity at $\epsilon = \infty$. If the singularity is a pole of order n, the function must then have the form

$$V_3 = V^{(n)}(\epsilon) = k\, i\left(\epsilon^n + \frac{1}{\epsilon^n}\right) \tag{2.15}$$

where k is a real constant. Hence

$$v_3 = \mathrm{Re}(V_3) = k\left(\frac{1}{R^n} - R^n\right) \sin n\theta \tag{2.16}$$

The corresponding W-functions are given by

$$W^{(1)}(\epsilon) = 2k \, \ln \epsilon \tag{2.17a}$$

$$W^{(2)}(\epsilon) = 4k\,(\epsilon - \frac{1}{\epsilon}) \tag{2.17b}$$

Thus for $n > 1$ the pressure distribution corresponding to $V^{(n)}$ gives an infinite force on a finite area, and the solution may be rejected because of this fact. However, the force corresponding to the solution $V^{(1)}$ is finite since $W^{(1)}$ only has a logarithmic singularity at the edge. Thus the only way to reject this solution is to make explicit use of the boundary condition (1.13'').

If one allows discontinuities across slits in the unit circle, poles of fractional order may be used. Let the wing correspond to a slit in the ϵ-plane. Consider the function

$$W = i(\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}}) \tag{2.18}$$

$$w = (\frac{1}{\sqrt{R}} - \sqrt{R}) \sin \frac{\theta}{2} \tag{2.19}$$

The corresponding downwash function is

$$V = \tfrac{1}{2}(\epsilon^{1/2} - \epsilon^{-1/2}) - \frac{1}{6}(\epsilon^{3/2} - \epsilon^{-3/2}) \tag{2.20}$$

$$v = \tfrac{1}{2}(\sqrt{R} - \frac{1}{\sqrt{R}}) \cos \frac{\theta}{2} - \frac{1}{6}\left(\sqrt{R^3} - \frac{1}{\sqrt{R^3}}\right)\cos \frac{3\theta}{2} \tag{2.21}$$

Evidently v vanishes on the wing $\theta = \pm\pi$ and on the unit circle $R = 1/R$. The pressure is infinite at the leading edge but gives a finite force for finite areas. Thus again it is possible to add this solution to the solution of Section II-B unless boundary condition (1.13'') is used.

It is interesting to note that Equations (2.18) and (2.20) have the symmetry properties of the lifting case. Thus it may be seen that in spite of the fact that the equation and the boundary conditions are antisymmetric in v, the resulting solution does not necessarily have this property unless uniqueness may be proved (Cf. Section I-D).

III. FLAT LIFTING WINGS OF ZERO THICKNESS

The general characteristics of flow fields induced by lifting wings of zero thickness have been discussed in Section I-B. When the shape of the wing is prescribed, finding the solution is difficult because the boundary conditions are mixed (involving both v and w). An unpleasant characteristic connected with a lifting wing is that it generally induces an upwash (positive or negative) in its own plane off the wing. Thus it is in general not possible to generate solutions by the very simple method that was used in Section II.

The prescribed shape gives the condition on v. When the surface is flat, this condition is v = constant. In this case, which is the only one considered in Section III, the condition on v may be eliminated. This important method (drawn from Busemann) will be discussed in connection with planform IIab. More general shapes (cambered profiles), both conical and nonconical, may be constructed from the wings introduced in Section VI.

Especially simple are those solutions which may be obtained from the solutions in Section II with the aid of the general principle to be described next.

A. Cases Reducible to Symmetrical Flow

Consider a conical symmetrical wing which may have any camber, but whose two edges are leading and supersonic. Then the Mach cone from the apex is disconnected by the wing into two separate regions without any contact. There is no contact between upper and lower surfaces around the edges since these edges are supersonic. Thus the flow above the wing is completely independent of the flow below the wing. Hence if the lower surface is kept unchanged but the upper surface is modified, the flow below the wing is unchanged. Modify the upper surface so that the wing changes from symmetrical to a wing of zero thickness. Then below the wing the flow is known from the symmetrical case. Next one obtains the flow above by the general symmetry rules for the lifting case given in Section I-B.

B. Planform Ia

This is the wide delta wing. Both edges are outside the Mach cone and leading. Hence the solution is obtained from Section II by the general principle just discussed. Let the wing have an angle of attack = α . Consider the solution for the symmetrical case where the half angle of the profile is = λ. Then the perturbation velocities below the wing in the lifting case are obtained from the formulas for the symmetrical case by replacing λ by α. To obtain the flow above the wing, λ will have to be replaced by $-\alpha$.

C. Planform Ib

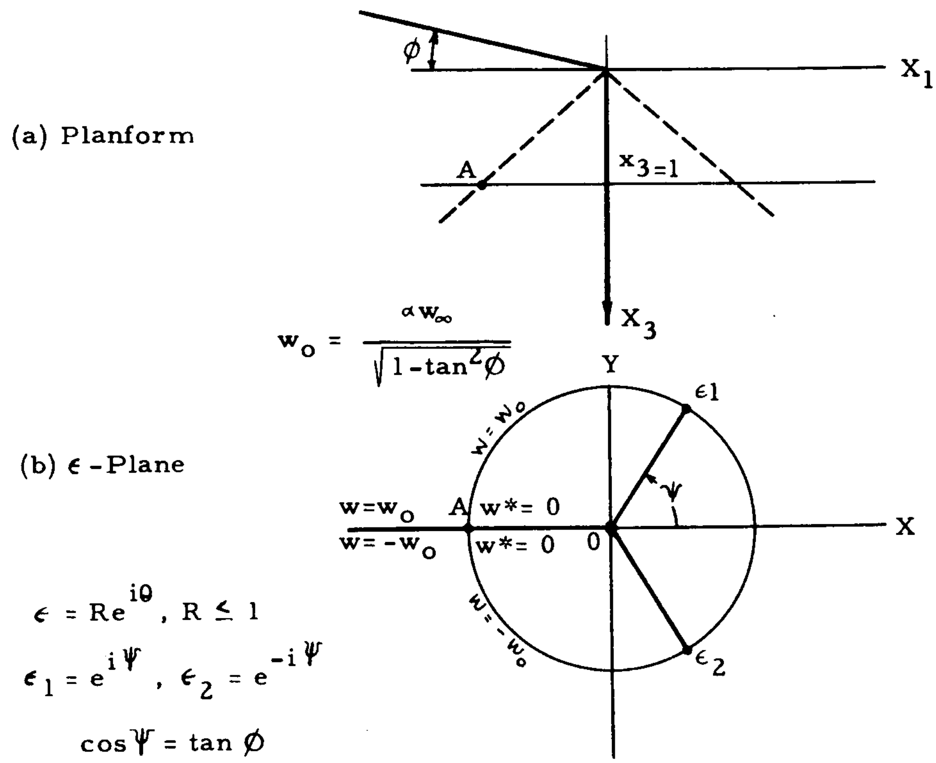
Here one edge is trailing and the wing is completely outside the Mach cone. In the applications such a case may occur at a wing tip if the rake angle is larger than the Mach angle. This case was discussed briefly by Busemann in Reference 4. Conditions on the wing are entirely supersonic and obtained by sweepback theory. For this special planform a flat wing is also a wing of constant-lift distribution. Within the Mach cone there is no lifting surface but a flow field (vortex field) induced by the lifting wing outside the cone. The solution is easily obtained by a superposition of two wings of constant-lift distribution with planform IIab (Cf. Ref. 4). Such wings will be considered in Section IV, where the superposition method will also be discussed.

D. Planform IIab: Pressure

This planform is of importance in considering the tip effect of an unyawed rectangular wing and applies also when the sweepback of the leading edge is different from zero but smaller than the complement of the Mach angle. The planform of the wing and the boundary conditions are illustrated in Figure 10. Busemann indicates how this case can be solved by using the principle of reflection. (He actually treats the slightly less general case $\phi = 0$). The solution will here be given in some detail in order to illustrate the use of certain very important principles.

As explained in Section I-K, w^* may be assumed to be zero on the flat wing AO.

Next use the principle of reflection as explained in Section II-B.



$\epsilon = Re^{i\theta}$, $R \leq 1$

$\epsilon_1 = e^{i\Psi}$, $\epsilon_2 = e^{-i\Psi}$

$\cos\Psi = \tan\phi$

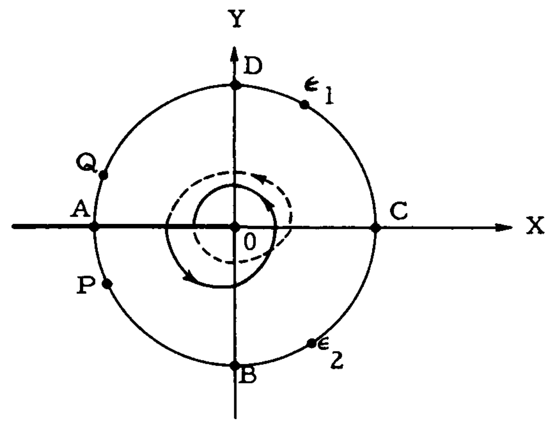
FIGURE 10. LIFTING WING: PLANFORM IIab

The function W is analytic inside the unit circle although not necessarily across the slit AO. The fact that $w^* = 0$ on this slit means in geometrical language that AO in the ϵ-plane is mapped on the real axis in the W-plane. Hence W may be continued analytically across the slit in such a way that points which are images with respect to AO are mapped on points which are images with respect to the line $w^* = 0$ in the W-plane. This fact may be expressed algebraically as follows (Cf. formula 2.9):

$$W(\overline{\epsilon}) = \overline{W(\epsilon)} \tag{3.1}$$

But Equation (3.1) implies that w is symmetric with respect to the X-axis, whereas it is known that in the physical problem w is antisymmetric. This apparent contradiction is resolved by introducing the Riemann surface. Start with a function W analytic in the unit circle except across OA, which represents the physical solution and is antisymmetric in W. If this solution is now continued across OA by reflection, one arrives at the lower sheet of the Riemann surface of the function W. It should be emphasized that any point may be reflected in OA, not only points directly above it. Thus the whole unit circle may be reflected in OA, and the image will form a second sheet of the Riemann surface which is a unit circle like the first sheet. The two sheets are joined along the slit OA. In Figure 11 a closed path around the origin is drawn. The function is analytic along the entire path. The fact that the path actually closes after two laps results from the assumption that $w^* = 0$ both on top and below the slit OA. If its value on the lower side of the slit were a constant different from zero, reflection would have given an infinite number of Riemann surfaces instead of just two.

The main advantage of extending the function is that the discontinuity OA has disappeared. On the Riemann surface the function is analytic across OA. Thus there are no boundary conditions in w^* or v, but only in w. The



Dotted Part of Contour is on the Lower Sheet

FIGURE 11. CLOSED PATH AROUND THE ORIGIN ON THE RIEMANN SURFACE

boundary is now a connected closed curve which represents the circumference of the two-unit circle. It runs like the closed curve in Figure 11. The values on the part of it which is on the upper sheet are given from physical considerations (Cf. Fig. 10). The values on the lower branch are given by reflection; e.g., at the point P in the physical plane $w = -w_o$ but on the corresponding point on the branch beneath it $w = w_o$ since that point is the image of a point Q in the physical plane where w was prescribed to be w_o.

This is the first example of Busemann's method for treating mixed problems, mentioned in Sections I-B and I-K. In the original mixed problem the boundary conditions are given in terms of both w and v. With the aid of Equation (1.45) the condition on v was transformed into a condition w^*.

Then with the aid of the principle of analytic extension by reflection, the condition on w* was replaced by conditions on w on the extended part of the Riemann surface. Although the domain obtained after the analytic extension is a double-sheeted circular disk, it is actually easier to handle than the original disk since it contains no cut across which w is discontinuous.

Now there is a standard way of unwinding such a double-sheeted surface, i.e., with the aid of the transformation

$$\nu = \sqrt{\epsilon} \tag{3.2}$$

This transformation maps the double-sheeted surface obtained by extending the ϵ-plane into a single-sheeted surface in the ν-plane, and the boundary value problem can now be formulated in the ν-plane. This fact is illustrated in Figure 12. Here C_1 is the transform of the point C on the upper sheet, C_2 the transform of the corresponding point on the lower surface, etc. In particular the wing OA appears in the imaginary axis in the ν-plane; the segment OA_1 represents the upper surface and OA_2 the lower surface.

W is analytic in the unit circle but has certain singularities on the circumference. These are of the same nature as those encountered for planform IIab in Section II. Thus the jump in w at the four points ν_i may be taken care of by appropriate logarithmic functions. This procedure leads to the following solution:

$$W(\epsilon) = + \frac{iw_o}{\pi} \ln \frac{(\nu - \nu_1)(\nu - \nu_4)}{(\nu - \nu_2)(\nu - \nu_3)} \tag{3.3}$$

$$\nu_1 = e^{\frac{i\psi}{2}}$$

$$\nu_2 = -e^{-\frac{i\psi}{2}}$$

$$\nu_3 = -e^{\frac{i\psi}{2}}$$

$$\nu_4 = e^{-\frac{i\psi}{2}}$$

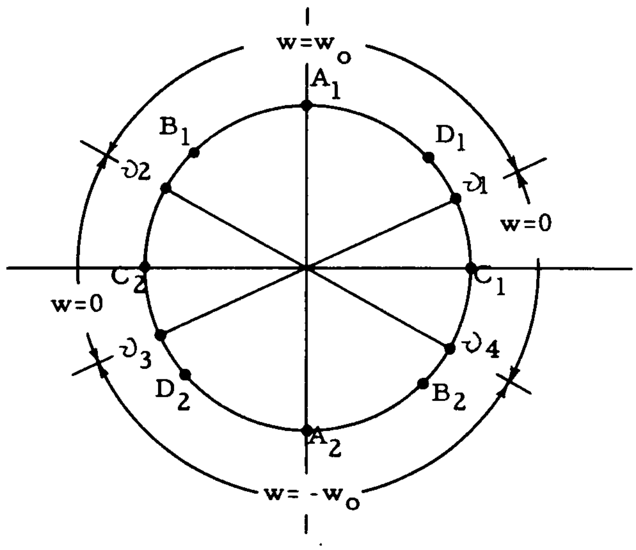


FIGURE 12. BOUNDARY CONDITIONS IN THE ν-PLANE

$\nu = \sqrt{\epsilon}$, ν_i as in Figure 12

$w = w_o$ on the top side of the wing outside Mach cone

$$w_o = \frac{+\alpha w_\infty}{\sqrt{1 - \tan^2\phi}}$$

It will follow from the evaluation of Equation (3.3) in physical coordinates that this solution actually satisfies all the boundary conditions. It is especially easy to evaluate w on the wing. However, in order to illustrate the technique, the general expression for w at any point inside the Mach cone will be derived.

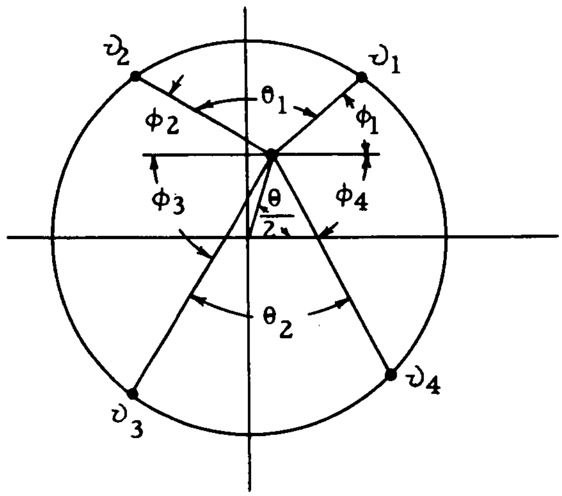
The following abbreviation will be used:

$$N = \sqrt{R} \tag{3.4}$$

From Figure 13 and formula (3.3):

$$\frac{\pi w}{w_o} = - \left[\phi_1 - (\pi - \phi_2) - (\pi + \phi_3) - \phi_4 + K \cdot 2\pi \right] \tag{3.5}$$

ν-Plane



All angles ϕ_i positive

$\nu_1 = e^{\frac{i\psi}{2}}$

FIGURE 13. NOTATION USED IN EQUATION (3.5)

By considering a point on the real axis of the ν-plane, it is seen that the integer k has to be given the value k = 1, because such a point corresponds in the physical plane to a point in the plane of the wing but not on the wing. Here w has to be zero because of antisymmetry and continuity. Thus

$$\frac{\pi w}{w_o} = - (\theta_2 - \theta_1) \tag{3.6}$$

where θ_2 and θ_1 are explained in Figure 13. Figure 13 also gives the relations

$$\left.\begin{aligned} \tan\phi_1 &= \frac{\sin\Psi/2 - N\sin\theta/2}{\cos\Psi/2 - N\cos\theta/2} \\ \tan\phi_2 &= \frac{\sin\Psi/2 - N\sin\theta/2}{\cos\Psi/2 + N\cos\theta/2} \\ \tan\phi_3 &= \frac{\sin\Psi/2 + N\sin\theta/2}{\cos\Psi/2 + N\cos\theta/2} \\ \tan\phi_4 &= \frac{\sin\Psi/2 + N\sin\theta/2}{\cos\Psi/2 - N\cos\theta/2} \end{aligned}\right\} \qquad (a)$$

From relation (a):

$$\left.\begin{aligned} \tan(\phi_1 - \phi_4) &= \frac{-2N\sin\theta/2\,(\cos\Psi/2 - N\cos\theta/2)}{1 + N^2\cos\theta - 2N\cos\Psi/2\cos\theta/2} \\ \tan(\phi_2 - \phi_3) &= \frac{-2N\sin\theta/2\,(\cos\Psi/2 + N\cos\theta/2)}{1 + N^2\cos\theta + 2N\cos\Psi/2\cos\theta/2} \end{aligned}\right\} \qquad (b)$$

From relation (b):

$$\tan(\phi_1 + \phi_2 - \phi_3 - \phi_4) =$$

$$= 2\,\frac{2N^3\cos\theta/2\cos\Psi/2\sin\theta - 2N\sin\theta/2\cos\Psi/2 - 2N^3\sin\theta/2\cos\Psi/2\cos\theta}{-(4N^2\sin^2\theta/2\cos^2\Psi/2) + N^4\sin^2\theta + 1 + N^4\cos^2\theta + 2N^2\cos\theta - 4N^2\cos^2(\theta/2)\cos^2\Psi/2}$$

$$= 2\,\frac{2N\sin\theta/2\cos\Psi/2\,(N^2-1)}{2N^2[\cos\theta - \cos\Psi] + N^4 - 2N^2 + 1} \qquad (c)$$

$$= \frac{4\sin\theta/2\cos\Psi/2}{2\dfrac{N}{N^2-1}\left[\cos\theta - \cos\Psi\right] + \dfrac{N^2-1}{N}}$$

By Equations (3.4) and (1.37i)

$$-\frac{N^2-1}{N} = \sqrt{\frac{2(1-r)}{r}} \qquad \text{(d)}$$

Also $\cos\psi = \tan\phi$

$$\cos\psi/2 = \sqrt{\frac{\cos 2(\psi/2)+1}{2}} = \sqrt{\frac{\tan\phi+1}{2}} \qquad \text{(e)}$$

Hence relation (c) may be written

$$\tan\left[(\phi_1+\phi_2)-(\phi_3+\phi_4)\right] = -\frac{\sin\frac{1}{2}\theta\sqrt{\tan\phi+1}\cdot 2\sqrt{2}}{\sqrt{\frac{2r}{(1-r)}}(\cos\theta-\tan\phi)+\sqrt{\frac{2(1-r)}{r}}} \qquad \text{(f)}$$

Simplifying Equation (f) and comparing with Equation (3.5) give the final formula:

$$w = \frac{\alpha w_\infty}{\pi\sqrt{1-\tan^2\phi}}\arctan\frac{2\sin\frac{1}{2}\theta\sqrt{(\tan\phi+1)r(1-r)}}{1-r(1-\cos\theta+\tan\phi)} \qquad (3.7)$$

This formula gives the value of w at a point (r, θ) inside the Mach cone if the value of w on the upper surface of the wing outside the Mach cone is $= +w_o = \frac{\alpha w_\infty}{\sqrt{1-\tan^2\phi}}$

By elementary trigonometry Equation (3.7) may be transformed to

$$w = \frac{\alpha w_\infty}{\pi\sqrt{1-\tan^2\phi}}\arccos\frac{1-r(1-\cos\theta+\tan\phi)}{\sqrt{1+r^2(\tan^2\phi-\sin^2\theta)-2r\cos\theta\tan\phi}} \qquad (3.7')$$

The advantage of the second form (3.7') is that on the wing itself $(\theta = \pi)$ it simplifies to

$$w = \frac{\alpha w_\infty}{\pi\sqrt{1-\tan^2\phi}}\arccos\frac{1-r(2+\tan\phi)}{1+r\tan\phi} \qquad (3.8)$$

or using the notation of Equations (1.35) and (1.36a)

$$w = \frac{\alpha W_\infty}{\pi\sqrt{1 - \tan^2\phi}} \text{ arc cos } \frac{1 + \tan\tau(2 + \tan\phi)}{1 - \tan\tau\tan\phi} \tag{3.8'}$$

Equation (3.8') is easily derived from Equation (3.6) by the above method if the cosine function is used instead of the tangent function.

It should be remembered that this formula is valid on the wing only and that hence τ takes only negative values. For points in the plane of the wing but not on the wing, τ is positive and w of course is 0 by anti-symmetry. Note that if the range of arc cos is restricted to angles between $-\pi$ and π, it is still a two-value function which for the same argument gives a positive and a negative value. The positive value corresponds to a point on the top surface of the wing.

For $\phi = 0$, Equation (3.8') specializes to the formula given by Busemann (Cf. Ref. 5).

E. Planform IIab: Downwash and Sidewash

The computations proceed along the same lines as those in Section II. First introduce the variable $\nu = \sqrt{\epsilon}$ into relations (1.42).

$$\frac{dU}{d\nu} = -\frac{1}{2}\left(\frac{1}{\nu^2} + \nu^2\right)\frac{dW}{d\nu} \tag{a}$$

$$\frac{dV}{d\nu} = -\frac{i}{2}\left(\frac{1}{\nu^2} - \nu^2\right)\frac{dW}{d\nu} \tag{b}$$

Then introduce the functions F and G by

$$dF = -\frac{\pi}{iw_o}\frac{1}{\nu^2}\,dW \tag{c}$$

$$dG = \frac{-\pi}{iw_o} \nu^2 \, dW \tag{d}$$

The following expansions into partial fractions will be needed:

$$\frac{1}{\nu^2(\nu - a)} = \frac{-1}{a^2\nu} - \frac{1}{a\nu^2} + \frac{1}{a^2(\nu - a)} \tag{e}$$

$$\frac{\nu^2}{\nu - a} = \nu + a + \frac{a^2}{\nu - a} \tag{f}$$

From Equation (3.3)

$$\frac{dW}{d\nu} = +\frac{iw_o}{\pi} \sum \pm \frac{1}{\nu - \nu_j} \tag{g}$$

where

+ is used for j = 1,4

and

- is used for j = 2,3

Combining relations (c), (e), (f), and (g)

$$\frac{-dF}{d\nu} = \sum \pm \left[-\frac{1}{\nu_j^2\nu} - \frac{1}{\nu_j\nu^2} + \frac{1}{\nu_j^2(\nu - \nu_j)} \right] \tag{h}$$

where + and - are used as in relation (g).

Since

$$\nu_1 = e^{i\frac{\Psi}{2}}, \quad \nu_2 = -e^{-i\frac{\Psi}{2}}, \quad \nu_3 = -e^{i\frac{\Psi}{2}}, \quad \nu_4 = e^{-i\frac{\Psi}{2}}$$

then

$$\sum \pm \frac{1}{\nu_j^2} = e^{-i\Psi} - e^{+i\Psi} - e^{-i\Psi} + e^{+i\Psi} = 0 \tag{i}$$

$$\sum_j \pm \frac{1}{\nu_j} = e^{-i\frac{\Psi}{2}} + e^{+i\frac{\Psi}{2}} + e^{-i\frac{\Psi}{2}} + e^{+i\frac{\Psi}{2}} = 2\left[e^{+i\frac{\Psi}{2}} + e^{-i\frac{\Psi}{2}}\right] = 4\cos\frac{\Psi}{2} \quad \text{(j)}$$

Hence

$$-F = +\frac{4}{\nu}\cos\frac{\Psi}{2} + \sum_j \pm \frac{1}{\nu_j^2}\ln(\nu-\nu_j) + \text{constant} \quad \text{(k)}$$

Similarly

$$\frac{-dG}{d\nu} = \sum \pm \left[(\nu+\nu_j) + \frac{\nu_j^2}{\nu-\nu_j}\right] = 4\cos\frac{\Psi}{2} + \sum_j \pm \frac{\nu_j^2}{\nu-\nu_j} \quad (\ell)$$

and

$$-G = 4\nu\cos\frac{\Psi}{2} + \sum \pm \nu_j^2 \ln(\nu-\nu_j) + \text{constant} \quad \text{(m)}$$

From relations (a) to (d), inclusive,

$$\frac{\pi U}{w_o} = +\frac{i}{2}(F+G) \quad \text{(n)}$$

$$\frac{\pi V}{w_o} = -\tfrac{1}{2}(F-G) \quad \text{(o)}$$

Thus finally

$$\frac{\pi U}{w_0} = -2i\left(\nu+\frac{1}{\nu}\right)\cos\frac{\Psi}{2} - i\cos\Psi \ln\frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)} \quad (3.9)$$

$$\frac{\pi V}{w_o} = -2\left(\nu-\frac{1}{\nu}\right)\cos\frac{\Psi}{2} - i\sin\Psi \ln\frac{(\nu-\nu_1)(\nu-\nu_2)}{(\nu-\nu_3)(\nu-\nu_4)} - \pi\sin\Psi \quad (3.10)$$

Note that according to Equations (1.44) and (1.45), $-w_o\cos\Psi = u_o$ and $-w_o\sin\Psi = v_o$ where u_o and v_o denote values of u and v on the top side of the wing outside the Mach cone. The choice of the constants of integration may be justified in the following way: The relations given above for u_o

and v_o are in particular valid at the point whose ν-coordinate is i. At that point Equations (3.9) and (3.10) are also valid. The first term in Equation (3.9) gives 0, and the real part of the second term is $-w_o \cos\Psi$, as can be seen by comparison with the corresponding term in Equation (3.3). But this is the correct value since $\cos\Psi = \tan\phi$. In Equation (3.10) it may be noticed that the first two terms are imaginary for $\nu = i$. Since $\sin\Psi = \sqrt{1-\cos^2\Psi} = \sqrt{1-\tan^2\phi}$, the constant agrees with the value from sweepback theory.

For the case of no sweepback ($\phi = 0, \Psi = \pi/2$), the simplified relations are obtained:

$$\frac{\pi U}{w_o} = -i\sqrt{2}\left(\nu + \frac{1}{\nu}\right) \tag{3.11}$$

$$\frac{\pi V}{w_o} = -\sqrt{2}\left(\nu - \frac{1}{\nu}\right) - i\ln\frac{(\nu-\nu_1)(\nu-\nu_2)}{(\nu-\nu_3)(\nu-\nu_4)} - \pi \tag{3.12}$$

From these relations it can be seen that there is flow around the side edge in the following sense. On the lower surface u = 0 on the wing outside the Mach cone. Inside the Mach cone it increases steadily and reaches the value $+\infty$ at the side edge. The positive sign means outward flow, as is physically reasonable since there is overpressure on the lower surface. At the edge, v (which of course is constant on the wing) jumps to the value $+\infty$ (positive sign means directed upward). On the upper surface the flow is inward, into the suction region. At the side edge $u = -\infty$ and increases monotonically to the value zero on the Mach cone. In the plane of the wing outside the wing, u has the constant value zero.

The sidewash and downwash velocities are obtained in terms of the physical coordinates r and θ and of the sweepback ϕ by taking the real parts of Equations (3.9) and (3.10), then making use of the trigonometric relations in Section III-D and relations (1.37).

Referring to Figure 13 and formula (3.9)

$$\frac{\pi u}{w_o} = 2\left(N - \frac{1}{N}\right)\cos\frac{\psi}{2}\cdot\sin\frac{\theta}{2} + \cos\psi\cdot\arg\left[\frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)}\right] \tag{3.13}$$

where

$$-\arg\left[\frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)}\right]$$

may be obtained from relation (f) in Section III-D . Hence the sidewash velocity inside the Mach circle is

$$\frac{\pi u}{w_o} = -2\left[\left(\frac{1-r}{r}\right)(1+\tan\phi)\right]^{\frac{1}{2}}\sin\frac{\theta}{2} - \tan\phi\cdot$$

$$\arc\cos\left\{\frac{1 - r(1-\cos\theta+\tan\phi)}{\left[1 + r^2(\tan^2\phi - \sin^2\theta) - 2r\cos\theta\tan\phi\right]^{\frac{1}{2}}}\right\} \tag{3.13'}$$

In the plane of the wing the formula for the sidewash is

$t > 0 : u = 0$

$$t < 0: \frac{\pi u}{w_o} = -2\left[\frac{1+t}{-t}(1+\tan\phi)\right]^{\frac{1}{2}} - \tan\phi \operatorname{arc\,cos}\frac{1+t(2+\tan\phi)}{1-t\cdot\tan\phi} \tag{3.13''}$$

That u = 0 off the wing in the plane of the wing follows directly from the discussion at the end of Section I-J.

Referring to Figure 13 and formula (3.10)

$$\frac{\pi v}{w_o} = -2\left(N - \frac{1}{N}\right)\cos\frac{\psi}{2}\cdot\cos\frac{\theta}{2} + \sin\psi\left\{\arg\left[\frac{(\nu-\nu_1)(\nu-\nu_2)}{(\nu-\nu_3)(\nu-\nu_4)}\right] - \pi\right\}$$

$$= 2\left[\left(\frac{1-r}{r}\right)(1+\tan\phi)\right]^{\frac{1}{2}}\cos\frac{\theta}{2} - \sin\psi\left\{\left[(\phi_2+\phi_3)-(\phi_1+\phi_4)\right] + \pi\right\} \qquad (3.14)$$

From relation (a) in Section III-D

$$\tan(\phi_2+\phi_3) = \frac{2\sin\frac{\psi}{2}\left(\cos\frac{\psi}{2} + N\cos\frac{\theta}{2}\right)}{(\cos\psi + N^2) + 2N\cos\frac{\theta}{2}\cdot\cos\frac{\psi}{2}} \qquad (p)$$

$$\tan(\phi_1+\phi_4) = \frac{2\sin\frac{\psi}{2}\left(\cos\frac{\psi}{2} - N\cos\frac{\theta}{2}\right)}{(\cos\psi + N^2) - 2N\cos\frac{\theta}{2}\cdot\cos\frac{\psi}{2}} \qquad (q)$$

$$\tan\left[(\phi_2+\phi_3)-(\phi_1+\phi_4)\right] = \frac{4N(1-N^2)\sin\frac{\psi}{2}\cos\frac{\theta}{2}}{4N^2\cos^2\frac{\theta}{2} - (1+2N^2\cos\psi + N^4)}$$

$$= \frac{2r\left[\left(\frac{1-r}{r}\right)(1-\tan\phi)\right]^{\frac{1}{2}}\cos\frac{\theta}{2}}{2r\cos^2\frac{\theta}{2} - r\tan\phi - 1} \qquad (r)$$

Hence the downwash velocity inside the Mach circle is

$$\frac{\pi v}{w_o} = 2\left[\left(\frac{1-r}{r}\right)(1+\tan\phi)\right]^{\frac{1}{2}}\cos\frac{\theta}{2}$$

$$+ (1-\tan^2\phi)^{\frac{1}{2}}\left\{\arctan\left[\frac{2\left[r(1-r)(1-\tan\phi)\right]^{\frac{1}{2}}\cos\frac{\theta}{2}}{1 + r\tan\phi - 2r\cos^2\frac{\theta}{2}}\right] - \pi\right\} \qquad (3.14')$$

For no sweepback, $\phi = 0$ (and hence $\psi = \pi/2$), Equations (3.13') and (3.14') reduce to

$$\frac{\pi u}{w_o} = -2\left(\frac{1-r}{r}\right)^{\frac{1}{2}} \sin\frac{\theta}{2} \tag{3.15}$$

$$\frac{\pi v}{w_o} = 2\left(\frac{1-r}{r}\right)^{\frac{1}{2}} \cos\frac{\theta}{2} + \arctan\left\{\frac{2\left[r(1-r)\right]^{\frac{1}{2}}\cos\frac{\theta}{2}}{1-2r\cos^2\frac{\theta}{2}}\right\} - \pi \tag{3.16}$$

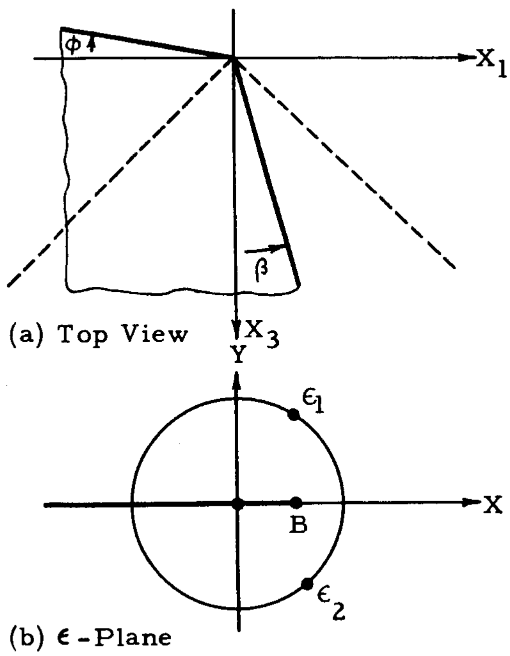


FIGURE 14. PLANFORM IIa

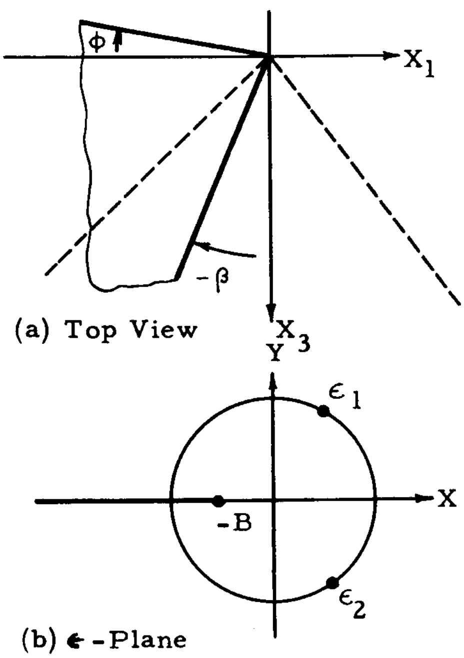


FIGURE 15. PLANFORM IIb

F. Planforms IIa and IIb

These planforms differ from planform IIab by the fact that the side edge is yawed ($\beta \neq 0$; Cf. Figs. 14 and 15). As pointed out in Sections

I-F and I-L, the solution may be obtained from that for planform IIab by a Lorentz transformation. One may use either a transformation of the velocity potential or a Lorentz transformation of the w-function. However, because of the different boundary conditions for leading and trailing edges, neither of these methods will give the complete solution for both types of planforms; some correction functions will be needed. For this reason both methods will be investigated. But first some preliminary formulas will be derived. Consider a wing as given in Figure 14 or 15. Such a wing may be unyawed by the oblique transformation (Cf. formula 1.17)

$$P \longrightarrow P^*$$

$$P = (x_1, x_2, x_3),\ P^* = \left(\frac{x_1 - bx_3}{\sqrt{1 - b^2}},\ x_2,\ \frac{x_3 - bx_1}{\sqrt{1 - b^2}} \right) \tag{3.17a}$$

$$b = \tan \beta$$

The corresponding homographic transformation is (Cf. Section I-L)

$$\epsilon \rightarrow \eta = \frac{\epsilon - B}{1 - B\epsilon},\ B = Ts(b) = \frac{1 - \sqrt{1 - b^2}}{b} \tag{3.17b}$$

Denote the sweepback of the original yawed wing by ϕ^b and that of the transformed wing by ϕ^o.

Then by Equation (1.50')

$$\tan \phi^o = \frac{\tan\phi^b - b}{1 - b \tan\phi^b} \tag{3.18a}$$

In the complex plane Equation (3.18a) will correspond to the formula:

$$\epsilon_1 = e^{i\psi^b} \rightarrow \eta_1 = e^{i\psi^o} = \frac{e^{i\psi^b} - B}{1 - Be^{i\psi^b}}, \; \cos\psi^o = \tan\phi^o \tag{3.19b}$$

Let w_o^b and w_o^o be the values of w on top of the wing outside the Mach cone in the yawed and the unyawed cases, respectively. Define u_o^b, u_o^o, v_o^b, v_o^o in a similar fashion. Then by the general formulas (1.44):

$$u_o^b = -w_o^b \tan\phi^b, \; u_o^o = -w_o^o \tan\phi^o$$

$$w_o^b = \frac{-v_o^b}{\sqrt{1-\tan^2\phi^b}}, \; w_o^o = \frac{-v_o^o}{\sqrt{1-\tan^2\phi^o}} \tag{3.20}$$

Using the fact that the inverse of Equation (3.18) is obtained by substituting -b for b (Cf. Eq. 1.20), one obtains from Equation (1.51)

$$\sqrt{1-\tan^2\phi^o} = \frac{\sqrt{1-b^2}\sqrt{1-\tan^2\phi^b}}{1 - b\tan\phi^b} \tag{3.21a}$$

$$\sqrt{1-\tan^2\phi^b} = \frac{\sqrt{1-b^2}\sqrt{1-\tan^2\phi^o}}{1 + b\tan\phi^o} \tag{3.21b}$$

If one also assumes that $v_o^o = v_o^b$, then Equations (3.20) and (3.21) imply

$$\frac{w_o^o}{u_o^b} = \frac{\sqrt{1-b^2}}{-b-\tan\phi^o} = \frac{1-B^2}{-2B-(1+B^2)\cos\psi^o}$$

$$= \frac{-1+b\tan\phi^b}{\tan\phi^b\sqrt{1-b^2}} = \frac{-(1+B^2)+2B\cos\psi^b}{(1-B^2)\cos\psi^b} \tag{3.22a}$$

$$\frac{w_o^o}{w_o^b} = \frac{\sqrt{1-b^2}}{1+b\tan\phi^o} = \frac{1-B^2}{1+B^2+2B\cos\psi^o}$$

$$= \frac{1-b\tan\phi^b}{\sqrt{1-b^2}} = \frac{1+B^2-2B\cos\psi^b}{1-B^2} \tag{3.22b}$$

G. Lorentz Transformation of the Velocity Potential; Solution for Planform IIa

This is the first method mentioned in Section I-L. Denote by U^o, V^o, and W^o the three functions defined by Equations (3.9), (3.10), and (3.3), respectively. According to Equation (1.57) one may form three new functions U^b, V^b, and W^b as follows (ν denoting $\sqrt{\eta} = \sqrt{\frac{\epsilon - B}{1 - B\epsilon}}$ rather than $\sqrt{\epsilon}$):

$$U^b(\epsilon) = \frac{1+B^2}{1-B^2} U^o(\eta) - \frac{2B}{1-B^2} W^o(\eta) = \frac{-2w_o^o}{\pi} \cdot \frac{1+B^2}{1-B^2} \cos\frac{\psi^o}{2}\, i\left(\nu + \frac{1}{\nu}\right)$$
$$+ \frac{i}{\pi} w_o^o \left[\frac{-(1+B^2)\cos\psi^o - 2B}{1-B^2} \ln \frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)} \right] \tag{3.23a}$$

$$V^b(\epsilon) = V^o(\eta) = \frac{-w_o^o}{\pi} 2\cos\frac{\psi^o}{2}\left(\nu - \frac{1}{\nu}\right) + \frac{v_o^o}{\pi}\left[i \ln \frac{(\nu-\nu_1)(\nu-\nu_2)}{(\nu-\nu_3)(\nu-\nu_4)} + \pi \right] \tag{3.23b}$$

$$W^b(\epsilon) = \frac{-2B}{1-B^2} U^o(\eta) + \frac{1+B^2}{1-B^2} W^o(\eta) = \frac{w_o^o}{\pi} \frac{4B}{1-B^2} \cos\frac{\psi^o}{2}\, i\left(\nu + \frac{1}{\nu}\right)$$
$$+ \frac{w_o^o}{\pi} \frac{2B\cos\psi^o + 1 + B^2}{1-B^2}\, i \ln \frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)} \tag{3.23c}$$

Since $v_o^o = v_o^b$ when the transformation is applied to the velocity potential (Cf. Eqs. 1.21c and 1.57b). Equation (3.22) is valid. With the aid of Equations (1.56), (3.20), and (3.22), Equations (3.23) may be simplified to

$$U^b(\epsilon) = \frac{u_o^b}{\pi}\left[c_1 i(\nu + \frac{1}{\nu}) + i\,\ell n\,\frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)}\right] \qquad (3.23a')$$

$$V^b(\epsilon) = \frac{v_o^b}{\pi}\left[c_2(\nu - \frac{1}{\nu}) + i\,\ell n\,\frac{(\nu-\nu_1)(\nu-\nu_2)}{(\nu-\nu_3)(\nu-\nu_4)} + \pi\right] \qquad (3.23b')$$

$$W^b(\epsilon) = \frac{w_o^b}{\pi}\left[c_3 i(\nu + \frac{1}{\nu}) + i\,\ell n\,\frac{(\nu-\nu_1)(\nu-\nu_4)}{(\nu-\nu_2)(\nu-\nu_3)}\right] \qquad (3.23c')$$

Here the constants c_i may be given in many forms; e.g.,

$$c_1 = \frac{\sqrt{2(1+\tan\phi^o)}}{b+\tan\phi^o} = \sqrt{\frac{2(1-b\tan\phi^b)(1+\tan\phi^b)}{1-b}}\cdot\frac{1}{(1+b)\tan\phi^b} \qquad (3.24a)$$

$$c_2 = \sqrt{\frac{2}{1-\tan\phi^o}} = \sqrt{\frac{2(1-b\tan\phi^b)}{(1+b)(1-\tan\phi^b)}} \qquad (3.24b)$$

$$c_3 = \frac{b\sqrt{2(1+\tan\phi^o)}}{1+b\tan\phi^o} = \frac{b}{1+b}\sqrt{\frac{2(1+\tan\phi^b)(1-b\tan\phi^b)}{1-b}} \qquad (3.24c)$$

The real velocity components corresponding to Equation (3.23) may be easily evaluated in physical coordinates. The velocity on the original wing is related to that on the unyawed wing by Equation (1.21c) and the velocity on the latter wing is given by Equations (3.8'), (3.13''), and (3.14'). It should be noted that according to Equation (1.21c), v is the same at corresponding points. Hence assuming w_∞ to be the same for both wings, α is also the same. Since the sweepback angles differ, w_o also takes on different values on the two wings according to Equation (3.20).

It follows from Section I that the three functions given by Equations (3.23) and (3.24) represent a conical flow field. However, one has to investigate whether they actually satisfy the boundary conditions for the wings in Figure 16. It follows immediately that all conditions on V are satisfied and that in the plane of the wing $w^* = 0$ on the wing and $w = 0$ off the wing. By construction these properties are transferred from the unyawed wing to the yawed wing. There remains to be checked the condition at the side edge. Formula (3.23) shows that for $B \neq 0$, W^b has here a pole of order $\frac{1}{2}$ because of the term $1/\nu$. A comparison with Equation (1.13') yields the result (Cf. also the discussion preceding Eq. 1.46):

Formulas (3.23) and (3.24) yield the correct solution for planform IIa (leading side edge) but not for IIb. (3.25)

A solution for planform IIb will be obtained from the second method of applying the oblique transformation.

E. Lorentz Transformation of the W-function; W for Planform IIb

In accordance with Equation (1.58) one generates a function $W_1^b(\epsilon)$ from the function W^o defined in Equation (3.3) by

$$W_1^b(\epsilon) = W^o(\eta)$$
$$\eta = \frac{\epsilon - B}{1 - B\epsilon} \tag{3.26}$$

As before the planform of the unyawed wing is given by Equation (3.18) or (3.19). Relations (3.20) and (3.21) are also valid. However, Equation (3.22) fails since the condition $v_o^o = v_o^b$ has to be replaced by $w_o^o = w_o^b$. Thus the angle of attack and v on the wing are different in the two cases. If v_o^b is given, then v_o^o and w_o^b are determined by

$$- w_o^o = - w_o^b = \frac{v_o^o}{\sqrt{1 - \tan^2\phi^o}} = \frac{v_o^b}{\sqrt{1 - \tan^2\phi^b}} \qquad (3.27)$$

which follows from Equation (1.44) and $w_o^o = w_o^b$.

According to Equation (3.21) one may write Equation (3.27) as

$$\frac{v_o^o}{v_o^b} = \frac{\sqrt{1 - b^2}}{1 - b\tan\phi^b} = \frac{1 + b\tan\phi^o}{\sqrt{1 - b^2}} \qquad (3.27')$$

If these values of v_o and w_o are used in determining W^o, then the function defined by Equation (3.26) satisfies the boundary conditions on the Mach circle for both planforms IIa and IIb. For $B \neq 0$, $\left(\frac{dW^b}{d\epsilon}\right)_{\epsilon=0} \neq 0$. Hence by Section I-L there is a logarithmic discontinuity in the downwash at the origin, and Equation (3.26) is good only for planform IIb. Furthermore, Equation (3.26) gives zero perturbation pressure at the edge ($\eta = 0$). This is a second reason why it gives the correct boundary condition for planform IIb but not for planform IIa.

In summarizing Section III-H above,

> W_1^b as defined in Equation (3.26) satisfies the boundary conditions for planform IIb. However, for planform IIa it fails for two reasons: First, there is not a pole in w at the side edge, and second, v has the correct value only for $-1 < t < 0$ and changes discontinuously at $t = 0$. (3.28)

One might, however, approach the solution of planform IIa in the following way. Start with the function defined in Equation (3.26). Then try to add a correction function which removes the difficulties mentioned in condition (3.28) without spoiling the boundary conditions which already are satisfied. This correction function must thus satisfy the following four conditions:

$$\left.\begin{array}{l} \text{1. } w = 0 \text{ on unit circle.} \\ \text{2. } w^* = 0 \text{ on the wing.} \\ \text{3. } W \text{ is regular inside the Mach circle except for a} \\ \quad \text{simple pole in } \nu \text{ at } \nu = 0. \\ \text{4. } v \text{ is zero for } -1 < t < 0\text{; at } t = 0 \text{ it has a dis-} \\ \quad \text{continuity which is the negative of that resulting} \\ \quad \text{from } W_1^b. \end{array}\right\} \tag{3.29}$$

Because of condition 1, W may be extended by reflection and is hence regular on the whole ν-plane except for the simple pole $1/\nu$ at the origin and the reflected pole ν at $\nu = \infty$. This fact, together with condition 2, leads to a function of the form

$$W = k\, W_2^b = k\, i\left(\nu + \frac{1}{\nu}\right) \tag{3.30}$$

where k is a real number and

$$\nu = \sqrt{\frac{\epsilon - B}{1 - B\epsilon}}$$

In order to satisfy condition 4, k will have to be determined from the equation

$$d(W_1^b + k\,W_2^b)_{\epsilon = 0} = 0 \tag{3.31}$$

If this equation is solved for k, one obtains (Cf. Eqs. 3.23c' and 3.24c)

$$k = \frac{w_o^b}{\pi} c_3$$
$$W^b = W_1^b + k\,W_2^b \tag{3.32}$$

where the function W^b is the one derived in Section III-G.

Returning to planform IIb, one sees that $W_1^b + c\,W_2^b$, c being any real constant, satisfies all boundary conditions except that for the side edge (1.13'). Hence for this planform there are infinitely many solutions unless a Kutta condition is imposed at the trailing edge. Thus Equation (1.13') is actually an essential boundary condition. In planform IIa the situation is different. Here the value of k is determined by the downwash, and the existence of a pole at the edge is proved to be necessary.

If poles of higher order are allowed, one might also consider the correction functions

$$W = ci(\nu^{-n} + \nu^{n}) \tag{3.33}$$

where n is an odd integer. These functions are excluded on the grounds that they yield infinite lift on a finite area (Cf. Section II-C).

I. Solution $W = i\left[\nu + (1/\nu)\right]$; Downwash and Sidewash for Planform IIb

The function $W_2^b = i\left[\nu + (1/\nu)\right]$ has already been discussed in this section. This function will appear in many connections and thus needs to be studied separately here. In computing the corresponding sidewash and downwash ($U_2{}^b$ and $V_2{}^b$), one has to distinguish between two cases: $\beta < 0$ and $\beta > 0$.

In the first, where $\beta < 0$, several properties of the flow field may be determined a priori. The solution represents a flat wing of zero thickness and of planform IIb. Although the angle of attack is everywhere zero, still the wing is lifting with an infinite pressure peak at the edge. On the Mach cone and outside the Mach cone all components of the perturbation velocity vanish. As pointed out in Section I-J, the sidewash u is = zero for $0 < t < 1$ and = constant = u_o for $b < t < 0$. This condition represents a vortex sheet behind the lifting wing. Since the lift is positive and u_o is the value on the upper side of this sheet, $u_o < 0$. The corresponding value on the lower side is $-u_o$. On the upper surface of the wing u increases from $-\infty$ to zero. At the edge the upwash is infinite and there is also a logarithmic infinity in V at t = 0 corresponding to the logarithmic jump of U.

Actual computation of $U_2{}^b$ and $V_2{}^b$ bears out these predictions. One obtains the following trio of functions:

$$U_2{}^b = -i\,\frac{B^2+1}{2B}\left(\nu+\frac{1}{\nu}\right) + \frac{(1-B)(1+B)^2}{4B\sqrt{-B}}\left[-i\,\ell n\,\frac{(\nu-\sqrt{-B})(\nu-\frac{1}{\sqrt{-B}})}{(\nu+\sqrt{-B})(\nu+\frac{1}{\sqrt{-B}})}+\pi\right] \tag{3.34a}$$

$$V_2{}^b = -\frac{1-B^2}{2B}\left(\nu-\frac{1}{\nu}\right) + \frac{(1-B)(1+B)^2}{4B\sqrt{-B}}\,\ell n\,\frac{(\nu-\sqrt{-B})(\nu+\frac{1}{\sqrt{-B}})}{(\nu+\sqrt{-B})(\nu-\frac{1}{\sqrt{-B}})} \tag{3.34b}$$

$$W_2{}^b = i\left(\nu+\frac{1}{\nu}\right) \tag{3.34c}$$

Here

$$B < 0, \quad \nu = \sqrt{\frac{\epsilon - B}{1 - B\epsilon}}$$

For evaluating Equation (3.34) in real coordinates, the following formulas may be used:

$$\frac{B^2+1}{2B} = \frac{1}{b}, \quad \frac{1-B^2}{2B} = \frac{\sqrt{1-b^2}}{b}, \quad \frac{(1-B)(1+B)^2}{4B\sqrt{-B}} = \frac{1+b}{b}\sqrt{\frac{1-b}{-2b}} \tag{3.35}$$

Also on the wing, putting $t^* = \frac{t - b}{1 - bt}$

$$i(\nu + \frac{1}{\nu}) = i\left[i\sqrt{|T^*|} + \frac{1}{i\sqrt{|T^*|}}\right] = \frac{1 - |T^*|}{\sqrt{|T^*|}} = \sqrt{\frac{2(1 - |t^*|)}{|t^*|}}$$

$$= \sqrt{\frac{2(1 + t^*)}{-t^*}} = \sqrt{\frac{2(1 - b)(1 + t)}{b - t}}$$

Hence

$$i(\nu + \frac{1}{\nu}) = \sqrt{\frac{2(1 - b)(1 + t)}{b - t}} \tag{3.36}$$

where

$$-1 < t < b$$

As may be seen by comparing Equations (3.23c), (3.28), and (3.22b), one may write the solution W_1^b for planform IIb as

$$W_1{}^b = W^b - c\, W_2{}^b \tag{3.37a}$$

where

$$c = \frac{w_o^o}{\pi} \frac{4B}{1 - B^2} \cos\frac{\psi^o}{2}$$

Hence the sidwash and downwash for planform IIb are obtained from Equation (3.34) and the formulas

$$U_1{}^b = U^b - cU_2{}^b \tag{3.37b}$$

$$V_1{}^b = V^b - cV_2{}^b \tag{3.37c}$$

The second case, $\beta > 0$, also represents a wing of zero thickness. However, since $b > 0$, the planform is IIa and the ray $t = 0$ is on the wing. The angle of attack is zero for $-1 < t < 0$ and jumps discontinuously to another constant value between $0 < t < b$. Wings with such discontinuities will be discussed in Section VI.

If the sidewash and downwash are computed, one obtains the following trio of functions corresponding to Equations (3.34)

$$U_2{}^b = -i\,\frac{B^2+1}{2B}\left(\nu+\frac{1}{\nu}\right) - \frac{(1-B)(1+B)^2}{4B\sqrt{B}}\,\ln\frac{(\nu - i\sqrt{B})(\nu + \frac{i}{\sqrt{B}})}{(\nu + i\sqrt{B})(\nu - \frac{i}{\sqrt{B}})} \tag{3.38a}$$

$$V_2{}^b = \frac{1-B^2}{2B}\left(\frac{1}{\nu}-\nu\right) + \frac{(1-B)(1+B)^2}{4B\sqrt{B}}\left[-i\ln\frac{(\nu - i\sqrt{B})(\nu - \frac{i}{\sqrt{B}})}{(\nu + i\sqrt{B})(\nu + \frac{i}{\sqrt{B}})} - \pi\right] \tag{3.38b}$$

$$W_2{}^b = i\left(\nu + \frac{1}{\nu}\right) \tag{3.38c}$$

As before, in Equation (3.35),

$$\frac{(1-B)(1+B)^2}{4B\sqrt{B}} = \frac{1+b}{b}\sqrt{\frac{1-b}{2b}}$$

Note that the second terms in the expressions for sidewash and downwash differ from the corresponding terms in Equation (3.34). The real part of the second term of Equation (3.38a) will now be evaluated for points on the wing:

$$\nu = i\sqrt{|T^*|}, \quad T^* = \frac{T - B}{1 - TB} < 0 \qquad \text{(a)}$$

Hence putting $|T^*| = R^*$

$$\frac{(\nu - i\sqrt{B})(\nu + \frac{i}{\sqrt{B}})}{(\nu + i\sqrt{B})(\nu - \frac{i}{\sqrt{B}})} = \frac{(\sqrt{R^*} - \sqrt{B})(\sqrt{R^*} + \sqrt{\frac{1}{B}})}{(\sqrt{R^*} + \sqrt{B})(\sqrt{R^*} - \frac{1}{\sqrt{B}})} = \frac{1 - \frac{\sqrt{R^*}}{1 - R^*} \cdot \frac{1 - B}{\sqrt{B}}}{1 + \frac{\sqrt{R^*}}{1 - R^*} \cdot \frac{1 - B}{\sqrt{B}}}$$

$$= \frac{1 - \sqrt{\frac{-t^*}{t^* + 1} \cdot \frac{1 - b}{b}}}{1 + \sqrt{\frac{-t^*}{t^* + 1} \cdot \frac{1 - b}{b}}} \qquad \text{(b)}$$

Here

$$t^* = \frac{t - b}{1 - bt} \qquad \text{(c)}$$

and

$$\frac{-t^*}{t^* + 1} = \frac{b - t}{(t+1)(1-b)} \qquad \text{(d)}$$

Expression (b) may also be transformed into

$$\left[\frac{\sqrt{b - t} - \sqrt{b(1 + t)}}{\sqrt{(1 + b)t}}\right]^2 \qquad \text{(e)}$$

Hence on the top surface of the wing ($-1 < t < b$) the following formulas result from Equation (3.38):

$$u = -\frac{1}{b}\sqrt{\frac{2(1-b)(1+t)}{(b-t)}} - \frac{2(1+b)}{b}\sqrt{\frac{1-b}{2b}}\;\ell n\left|\frac{b-t-\sqrt{b(1+t)}}{\sqrt{(1+b)t}}\right| \tag{3.39a}$$

$$v = 0 \text{ for } -1 < t < 0.$$

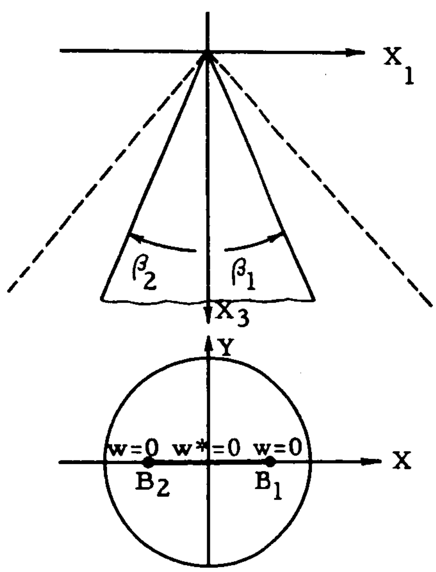
$$v = -\pi\frac{1+b}{b}\sqrt{\frac{1-b}{2b}} \text{ for } 0 < t < b \tag{3.39b}$$

$$w = \sqrt{\frac{2(1-b)(1+t)}{b-t}} \tag{3.39c}$$

J. Planform III

Consider a flat lifting delta wing with both edges inside the Mach cone. To begin with, leave unspecified whether one edge is trailing (planform IIIb) or whether both are leading (planform IIIa) and what the nature of the flow at the edges is. The boundary conditions are shown in Figure 16.

(a) Planform



(b) ϵ-Plane

FIGURE 16. PLANFORM III

The Riemann surface of the function W can easily be obtained by reflection (Cf. Sections II-B and III-B). The fact that $w = 0$ on the unit circle means in geometrical language that W maps the unit circle in the ϵ -plane on the imaginary axis in the W-plane. Then by analytic continuation one can define W outside the unit circle in such a way that points which are images with respect to the unit circle are mapped on points which are images with respect to the imaginary axis in the W-plane. The image of ϵ in the unit circle is $1/\bar{\epsilon}$ and the image of $w + iw^*$ in the imaginary axis is $-w + iw^*$. Hence the rule for extension is in algebraic formulation (Cf. Eq. 2.9)

$$W(1/\bar{\epsilon}) = -\overline{W(\epsilon)} \tag{3.40}$$

Before extension, W was analytic inside the unit circle except across the slit B_2B_1 and except for possible poles at B_1 and B_2. By extension W will then be analytic in the outside of the unit circle except for the slit ($1/B_1$ to $+\infty$, $-\infty$ to $1/B_2$), which is the image of the slit inside the circle. Hence W is defined on the whole Riemann sphere (ϵ -plane and point at infinity) except for the two slits mentioned. By Equation (1.46), $w^* = 0$ on the slit B_2B_1 both on the top side and the bottom side. By Equation (3.40), this condition is then also true for the reflected slit. Since $w^* = 0$, one may again use analytic extension for continuing the function across the slits. The rule for this extension will be

$$W(\bar{\epsilon}) = \overline{W(\epsilon)} \tag{3.41}$$

According to this rule $w = \mathrm{Re}(W)$ is symmetric with respect to X-axis. But from considerations in Section I it is known that in the part corresponding to physical reality w is antisymmetric with respect to the wing. Hence continuation across the slits will not give the values of W that the original part of the function assumed but will lead to a new sheet of the Riemann surface The situation is very similar to that of planform IIab except that only one slit had to be considered in planform IIab. It may again be concluded that the Riemann surface has only two sheets because $w^* = 0$ on both sides of both slits. Hence the four possible reflections (in the upper and lower parts of the two slits) yield the same result, and two successive reflections give back to the original function (Cf. planform IIab). Thus the Riemann surface consists of two spheres joined along the two slits B_2B_1 and $\frac{1}{B_1}\infty\frac{1}{B_2}$. This condition is shown in Figure 17 where some closed curves have been drawn to illustrate the nature of the branch points.

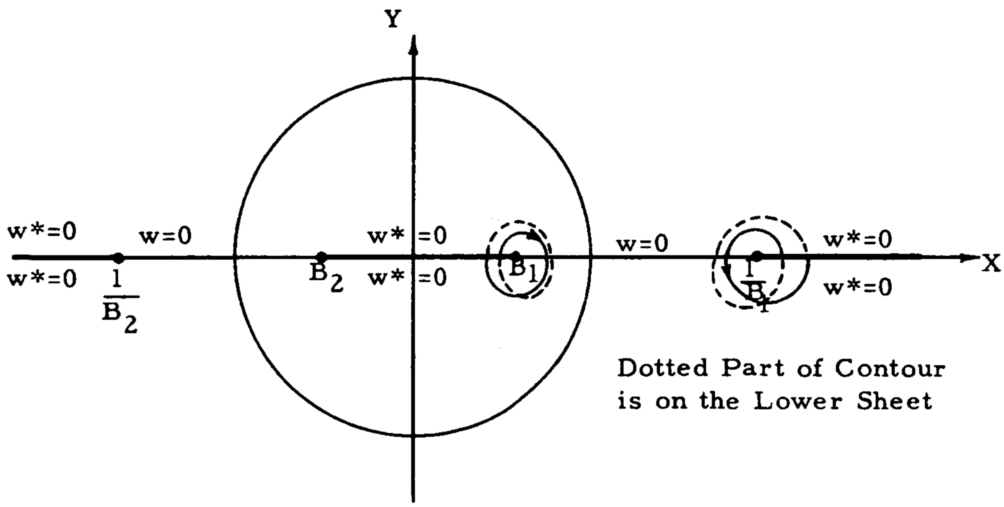


FIGURE 17. RIEMANN SURFACE FOR PLANFORM III

This is a well-known Riemann surface (Cf. pp. 229 and 244 ff. of Ref. 18; also Part III, Chap. 9.1 of Ref. 19). It has the connectivity of a torus, and any function which is meromorphic on this surface (i.e., is analytic except for isolated poles) may be expressed as a rational function of the two quantities ϵ and the radical

$$\rho = \sqrt{(\epsilon - B_1)(\epsilon - B_2)(\epsilon - \frac{1}{B_1})(\epsilon - \frac{1}{B_2})} \tag{3.42}$$

Furthermore, for any meromorphic function on a compact Riemann surface, the number of zeros equals the number of poles; when the poles and zeros are known, the rational function of ρ and ϵ can be written immediately. From the process of extension just described, it follows that the only possible poles are at the end points of the slits.

First consider a basic solution which has a simple pole at B_2 and a zero at B_1. The connection with physical condition at the two edges will be discussed later in this section. From Equation (3.40) one concludes that $1/B_2$ must be a simple pole and $1/B_1$ a zero of the function W. Since no other poles are possible and the number of zeros already equals the number of poles, it may be concluded that there are no other zeros either. The rational function of ρ and ϵ which has the correct zeros and poles is easily seen to be

$$W = k \frac{(\epsilon - B_1)(\epsilon - \frac{1}{B_1})}{\rho} \tag{3.43}$$

where k is a real constant. The constant k has to be real but is otherwise arbitrary. Then for real values of ϵ, Equation (3.43) gives the correct values: $w^* = 0$ on the slits and $w = 0$ between the slits. Furthermore, $w = 0$ on the unit circle. Hence Equation (3.43) satisfies all boundary conditions except Equation (1.4'), which will be used in determining k.

Note that a simple pole does not have the form

$$\frac{1}{\epsilon - B_2} \quad \text{but} \quad \frac{1}{\sqrt{\epsilon - B_2}}$$

This condition exists because B_2 is a branch point, and a neighborhood of B_2 on the Riemann surface cannot be described by the variable ϵ; instead one needs a uniformizing variable (Cf. pp. 214 ff. of Ref. 18) like $\sqrt{\epsilon - B_2}$. By a similar reasoning one sees that $(\epsilon - B_1)$ represents a zero of order 2. However, in Equation (3.43) $\sqrt{\epsilon - B_1}$ appears in the denominator. If this factor is divided out, $\sqrt{\epsilon - B_1}$ remains in the numerator and represents a simple zero. To show the poles and zeros, write

$$W = k \sqrt{\frac{-(\epsilon - B_1)(B_1 \epsilon - 1)}{(\epsilon - B_2)(B_2 \epsilon - 1)}} \tag{3.43'}$$

where k is a real number.

Next look for solutions with simple poles at B_1 and B_2. Such a solution is found by adding to Equation (3.43') a similar solution with B_1 and B_2 interchanged:

$$W = k_1 \sqrt{\frac{-(\epsilon - B_1)(B_1 \epsilon - 1)}{(\epsilon - B_2)(B_2 \epsilon - 1)}} + k_2 \sqrt{\frac{-(\epsilon - B_2)(B_2 \epsilon - 1)}{(\epsilon - B_1)(B_1 \epsilon - 1)}} \tag{3.44}$$

where k_1 and k_2 are real numbers. Evidently Equation (3.44) satisfies the boundary conditions for W since each of the terms does and the conditions are homogeneous: Each of the terms has the property that its real term vanishes on the unit circle and its imaginary term vanishes on the real axis between B_1 and B_2. Hence the sum of the two terms has the same property. Furthermore, since there are poles of the right order both at B_1 and B_2, there remains only one thing to be checked, namely, the absence of a logarithmic singularity at $\epsilon = 0$ if this point is on the wing (planform IIIa). This can be achieved by choosing k_1 and k_2 in the correct ratio so that $\left(\frac{dW}{d\epsilon}\right)_{\epsilon = 0}$ vanishes (Cf. Section I-J and the use of the correction function in Section III-H). Then

$$\left(\frac{dW}{d\epsilon}\right)_{\epsilon = 0} = \frac{k_1}{2}\sqrt{\frac{-B_2}{B_1}}\ \frac{-B_2(1 + B_1^2) + B_1(1 + B_2^2)}{-B_2^2} \tag{a}$$

$$+ \frac{k_2}{2}\sqrt{\frac{-B_1}{B_2}}\ \frac{-B_1(1 + B_2^2) + B_2(1 + B_1^2)}{-B_1^2} = 0 \text{ or } \frac{k_2}{k_1} = \frac{B_1}{-B_2} \tag{b}$$

Hence for planform IIIa, Equation (3.44) must be of the form

$$W = k'\left[B_2\sqrt{\frac{-(\epsilon - B_1)(B_1\epsilon - 1)}{(\epsilon - B_2)(B_2\epsilon - 1)}} - B_1\sqrt{\frac{-(\epsilon - B_2)(B_2\epsilon - 1)}{(\epsilon - B_1)(B_1\epsilon - 1)}}\right] \tag{3.45}$$

where k' is a real number.

For the unyawed delta wing $\beta_1 = -\beta_2 = \beta$ and $B_1 = -B_2 = B$. Then

$$\left.\begin{aligned} W &= k'B\left[\frac{-\left[B\epsilon^2 - (1 + B^2)\epsilon + B\right] + \left[-B\epsilon^2 - (1 + B^2)\epsilon - B\right]}{\sqrt{(\epsilon^2 - B^2)(B^2\epsilon^2 - 1)}}\right] \\ &= \frac{-2k'B^2(1 + \epsilon^2)}{\sqrt{(\epsilon^2 - B^2)(B^2\epsilon^2 - 1)}} \end{aligned}\right\} \tag{c}$$

Hence for the unyawed case

$$W = k'' \frac{B(\epsilon^2 + 1)}{\sqrt{(\epsilon^2 - B^2)(B^2\epsilon^2 - 1)}} \tag{3.46}$$

where $k'' = -2k'B$ is a real number.

This is the formula obtained by Stewart in Reference 16. It may also be obtained directly by the following reasoning. The Riemann surface for W is as described in connection with Figure 17. There are four poles, namely, at $\pm B$, and $\pm 1/B$ and these are the only singularities. Thus the function may be written down as a rational function of ϵ and ρ (Eq. 3.42) as soon as the four zeros are found. Now $w = 0$ on unit circle. Furthermore, for the unyawed wing the imaginary axis is a line of symmetry. Hence $\partial w/\partial x = 0$ here and thus also $\partial w^*/\partial y = 0$. But $w^* = 0$ at $\epsilon = 0$. Hence $w^* = 0$ on the whole imaginary axis. Thus at $\pm i$, $w = w^* = 0$ and also $W = w + iw^* = 0$. But these are actually four zeros, since the surface is double-sheeted and the points $\epsilon = \pm i$ occur on both sheets. Thus the information about the zeros and the poles is complete, and this fact yields Equation (3.46) immediately. Comparing the results with the boundary condition (1.13') for the edges, one reaches the conclusion that Equation (3.43') is the solution for planform IIIb if B_1 corresponds to the trailing edge and (Eq. 3.45) is the solution for planform IIIa. Both formulas contain an undetermined constant k which will have to be determined from the angle of attack (Cf. Eqs. 3.50 to 3.54, inclusive).

As for planform IIb, one obtains the result that if a pole is allowed at a trailing edge there are infinitely many solutions for planform IIIb, because the angle of attack gives just one relation between k_1 and k_2 in Equation (3.44). Thus this formula represents a one-parameter family of solutions for planform IIIb, all of which have poles at the trailing edge except when $k_2 = 0$ (Cf. Ref. 16 for a discussion on higher-order singularities).

To evaluate w in physical coordinates in the plane of the wing, observe that

$$\frac{(\epsilon - B_1)(B_1\epsilon - 1)}{(\epsilon - B_2)(B_2\epsilon - 1)} = \frac{\frac{\epsilon^2 + 1}{2\epsilon} - \frac{B_1^2 + 1}{2B_1}}{\frac{\epsilon^2 + 1}{2\epsilon} - \frac{B_2^2 + 1}{2B_2}} \cdot \frac{B_1}{B_2} = \frac{\frac{1}{t} - \frac{1}{b_1}}{\frac{1}{t} - \frac{1}{b_2}} \cdot \frac{B_1}{B_2} \qquad \text{(d)}$$

Hence for the wing with one edge trailing

$$w = k\sqrt{\frac{B_1}{B_2}} \cdot \sqrt{-\frac{1 - \frac{t}{b_1}}{1 - \frac{t}{b_2}}} \qquad (3.47)$$

For the wing with two leading edges

$$w = -k'\sqrt{-B_1 B_2}\left[\sqrt{\frac{1 - \frac{t}{b_1}}{1 - \frac{t}{b_2}}} + \sqrt{\frac{1 - \frac{t}{b_2}}{1 - \frac{t}{b_1}}}\right] \qquad (3.48)$$

Finally, for the symmetrical case ($b_1 = -b_2 = b$), Equation (3.48) reduces to

$$w = -k'B \frac{2}{\sqrt{1 - (\frac{t}{b})^2}} = \frac{k''}{1 - (\frac{t}{b})^2} \qquad (3.49)$$

To complete the solution the constant k has to be determined as a function of the angle of attack. One way of finding k is to compute the downwash from the pressure and to compare it with the known values on the wing. Calculations have been carried out for the symmetrical case by Stewart (Cf. Ref. 16). For the yawed case the constants were first found in Reference 20.

For planform IIIb the constant k is given by (see Ref. 20)

$$k\sqrt{\frac{B_1}{B_2}} = \frac{\alpha\, w_\infty \sqrt{b_1 b_2}}{E(C) + C_1 D \Pi_o\left[C,(-D_1^2)\right]} \tag{3.50}$$

where

$$A = \sqrt{\frac{1 - b_1 b_2 + \sqrt{1 - b_1^2}\sqrt{1 - b_2^2}}{2}}$$

$$C = \frac{1}{A}\sqrt[4]{(1 - b_1^2)(1 - b_2^2)}$$

$$C_1 = \sqrt{1 - C^2}$$

$$D = \frac{b_1 + b_2}{1 + b_1 b_2 + \sqrt{1 - b_1^2}\sqrt{1 - b_2^2}}$$

$$D_1 = \sqrt{1 - D^2}$$

$E(C)$ = the elliptic integral of the first kind

$\Pi_o\left[C,(-D_1^2)\right]$ = the elliptic integral of the second kind

$$= \int_o^1 \frac{d\lambda}{(1 - \lambda^2 D_1^2)\sqrt{(1 - \lambda^2)(1 - C^2\lambda^2)}}$$

The corresponding formula for planform IIIa is

$$k' = \frac{-\alpha w_{\infty}}{1 - B_1 B_2} \frac{1}{E(C)} \tag{3.51}$$

where C is defined as in (3.50).

In the symmetrical case Equation (3.51) reduces to

$$k'' = -2k' B = \frac{\alpha w_{\infty} \cdot 2B}{1 + B^2} \cdot \frac{1}{E\left[\frac{1 - B^2}{1 + B^2}\right]} = \frac{\alpha w_{\infty} b}{E(\sqrt{1 - b^2})} \tag{3.52}$$

From Equations (1.43a), (3.49), and (3.52) it follows easily that the sidewash on the wing has the simple form

$$u = \frac{-\alpha w_{\infty} (t/b)}{E(\sqrt{1 - b^2})\sqrt{1 - (t/b)^2}} \tag{3.53}$$

With the aid of this formula and the rule for oblique transformation (Eq. 1.21), the constant for the yawed case of planform IIIa (Eq. 3.51) may be deduced from the unyawed case (Eq. 3.53).

The downwash is determined from

$$\frac{dV}{d\epsilon} = -i\alpha w_{\infty} \cdot \frac{(1 + B^2)}{E\left[\frac{1 - B^2}{1 + B^2}\right]} \frac{(1 - \epsilon^2)^2}{\left[(\epsilon^2 - B^2)(B^2\epsilon^2 - 1)\right]^{3/2}} \tag{3.54}$$

IV. WINGS OF CONSTANT-LIFT DISTRIBUTION

The wings considered in this section are lifting wings of zero thickness. In this respect they are similar to the wings discussed in Section III. Thus the local angle of attack is the same at corresponding points on the top surface and on the bottom surface, and v is symmetric and u and w are antisymmetric (Cf. Section I-B). However, in Section III a constant value of v was prescribed on the wing since the shape of the wing was given. In the present case the lift of the wing is prescribed and hence w is given. It will always be assumed that the lift distribution is constant. Thus on the top surface $w = \text{constant} = w_o$ and on the bottom surface $w = -w_o$. For the purpose of obtaining solutions the present case is thus much closer to Section II than to Section III, because, in the present case, w is prescribed in the whole plane of the wing and also of course on the Mach cone. Thus there are sufficient boundary conditions for one perturbation velocity. As for the symmetrical wings of given shape (Cf. Section II), this fact makes the solution straightforward. Another similarity to Section II is that interference between two wings of constant lift in the same plane does not affect the boundary conditions, for there is only downwash and sidewash interference but no pressure interference between two wings of the constant-lift type.

These general characteristics simplify the solution in two ways. First, because of the last-mentioned property it is enough to obtain the solution for some very simple planforms and then obtain the other planforms by simple superposition as in Section II. By making a slight modification of the procedure in Section II planform II will be chosen for the basic solutions. Planforms I and III are then easily obtained by superposition. As a matter of fact if, e.g., solutions for planform IIb have been found, one may obtain solutions for planform IIa by subtracting from the two-dimensional case. Condition (1.13) is not of importance for the deduction of the analytic formulas since for the constant-lift wings there is no difference in principle between a leading and a trailing side edge. But such a subtraction is very useful for actual numerical computations.

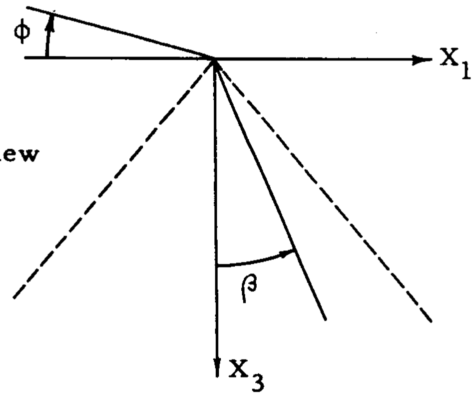
The second simplification is that for solving problems in this section one may merely borrow the solutions from Section II. More precisely, if a certain function $f(\epsilon)$ is the correct solution for V for a symmetrical wing with $v = \pm a$ prescribed on the wing, then the same function $f(\epsilon)$ is a correct solution for W for a wing of the same planform with $w = \pm a$ on the wing. Actually $W(\epsilon)$ will be obtained for a wing of planform IIab by this procedure; W for the general planform II is then obtained by a homographic transformation. This transformation is completely trivial for wings with prescribed lift distribution, and none of the complications of Section III arises. However, because of the special use of the wings in this section (Cf. Section IV-B), the interest is mainly in the downwash and the sidewash, which, of course, may not be taken immediately from Section II.

Thus, the program is as follows: From Section II one obtains immediately the function representing W for planform II. From this function V and U are computed with the aid of Equation (1.42), and v and u are then evaluated in terms of physical coordinates. To evaluate w separately is of course superfluous since this function may be had immediately from Section II.

A. Planform II

This planform will be treated in detail because of its importance for downwash calculations. The boundary conditions are shown in Figure 18. Since the lift distribution is constant on the wing, w is also constant on the wing by Equation (1.3a). It is assumed that the value of w on the top side of the wing is w_o. Hence it is $-w_o$ on the bottom side.

(a) Top View



(b) ϵ -Plane

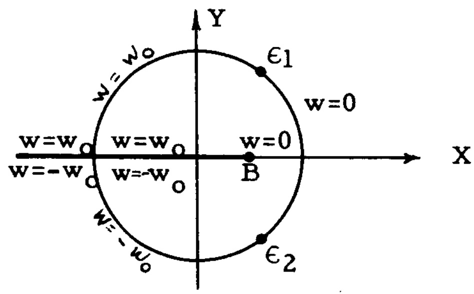


FIGURE 18. WING OF CONSTANT-LIFT DISTRIBUTION, PLANFORM II

As in Section III-F the following abbreviations have been introduced:

$$b = \tan\beta, \; B = \frac{1 - \sqrt{1 - b^2}}{b}, \; \cos\Psi = \tan\phi \tag{4.1}$$

$$\epsilon_1 = e^{i\Psi}, \; \epsilon_2 = \overline{\epsilon}_1$$

As before, move the point B, representing the edge, to the origin by the transformation

$$\eta = \frac{\epsilon - B}{1 - B\epsilon} \tag{4.2}$$

The singularities ϵ_1 and ϵ_2 are thereby moved to η_1 and η_2, respectively, where

$$\eta_1 = \frac{\epsilon_1 - B}{1 - B\epsilon_1} \text{ and } \eta_2 = \frac{\epsilon_2 - B}{1 - B\epsilon_2} \tag{4.3}$$

In the η-plane the boundary conditions look as shown in Figure 19.

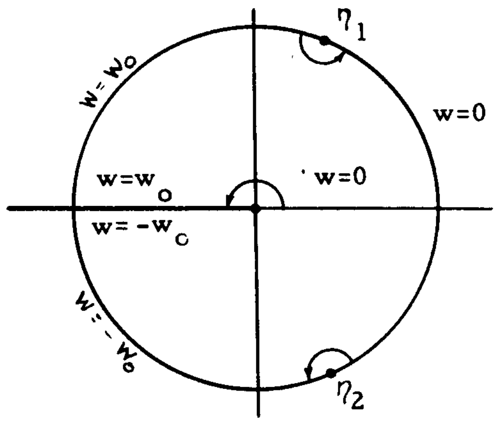


FIGURE 19. BOUNDARY CONDITIONS IN THE η-PLANE

As is seen, there are three logarithmic singularities inside the unit circle, namely, at $\eta = 0$, η_1, η_2. Figure 19 shows paths consisting of small half circles around the singularities. The paths are inside the circle and do not cross the slit on the negative real axis. If η varies counterclockwise on these paths, $\frac{w}{w_o}$ increases by 1 at $\eta = 0$ and decreases by 1 at

η_1 and η_2. A function which has these singularities (Cf. Ref.4) is

$$W = \frac{iw_o}{\pi}\left[\ell n(\eta - \eta_1) + \ell n(\eta - \eta_2) - \ell n\,\eta\right] \tag{4.4}$$

On the real axis of the η-plane the real parts of the first two terms represent angles of equal magnitude but opposite sign. Hence the only term that gives a contribution to w = Re(W) is $\frac{-iw_o}{\pi}\,\ell n\,\eta$ which gives the correct values $w = +w_o$, $-w_o$, or 0.

For the applications, the components u and v rather than w are of interest. Explicit formulas for these components will be derived from Equations (4.4) and (1.42). As in Section III-E the functions F and G are introduced by

$$dF = \frac{-\pi}{iw_o}\,\frac{dW}{\epsilon} \tag{a}$$

and

$$dG = \frac{-\pi\epsilon}{iw_o}\,dW \tag{b}$$

The following auxiliary formulas will be needed:

$$\frac{d\eta}{d\epsilon} = \frac{(1 - B\epsilon) - (\epsilon - B)(-B)}{(1 - B\epsilon)^2} = \frac{1 - B^2}{(1 - B\epsilon)^2} \tag{c}$$

$$\frac{1}{\eta - \eta_1} = \frac{1}{\frac{\epsilon - B}{1 - B\epsilon} - \frac{\epsilon_1 - B}{1 - B\epsilon_1}} = \frac{(1 - B\epsilon)(1 - B\epsilon_1)}{\epsilon - \epsilon_1 - B^2(\epsilon - \epsilon_1)}$$

$$= \frac{(1 - B\epsilon)(1 - B\epsilon_1)}{(1 - B^2)(\epsilon - \epsilon_1)} \tag{d}$$

$$\frac{dW}{d\epsilon} = \frac{iw_o}{\pi}\left[\frac{1 - B\epsilon_1}{(1 - B\epsilon)(\epsilon - \epsilon_1)} + \frac{1 - B\epsilon_2}{(1 - B\epsilon)(\epsilon - \epsilon_2)} - \frac{(1 - B^2)}{(\epsilon - B)(1 - B\epsilon)}\right] \tag{e}$$

$$\frac{1}{\epsilon} \cdot \frac{1 - B\epsilon_1}{(1 - B\epsilon)(\epsilon - \epsilon_1)} = -\frac{1 - B\epsilon_1}{\epsilon_1} \cdot \frac{1}{\epsilon} + \frac{1}{\epsilon_1} \cdot \frac{1}{\epsilon - \epsilon_1} + \frac{B^2}{1 - B\epsilon} \tag{f}$$

$$\frac{1 - B^2}{\epsilon(\epsilon - B)(1 - B\epsilon)} = -\frac{1 - B^2}{B\epsilon} + \frac{1}{B(\epsilon - B)} + \frac{B^2}{1 - B\epsilon} \tag{g}$$

$$\frac{\epsilon(1 - B\epsilon_1)}{(1 - B\epsilon)(\epsilon - \epsilon_1)} = \frac{\epsilon_1}{\epsilon - \epsilon_1} + \frac{1}{1 - B\epsilon} \tag{h}$$

$$\frac{\epsilon(1 - B^2)}{(\epsilon - B)(1 - B\epsilon)} = \frac{B}{\epsilon - B} + \frac{1}{1 - B\epsilon} \tag{i}$$

Hence integrating formulas (a) and (b) and neglecting constants of integration:

$$\begin{aligned} -F = &-\frac{1 - B\epsilon_1}{\epsilon_1}\,\ln\epsilon + \frac{1}{\epsilon_1}\ln(\epsilon - \epsilon_1) - B\ln(1 - B\epsilon) \\ &-\frac{1 - B\epsilon_2}{\epsilon_2}\,\ln\epsilon + \frac{1}{\epsilon_2}\ln(\epsilon - \epsilon_2) - B\ln(1 - B\epsilon) \\ &-\left[-\frac{1-B^2}{B}\ln\epsilon + \frac{1}{B}\ln(\epsilon - B) - B\ln(1 - B\epsilon)\right]. \end{aligned} \tag{j}$$

$$\begin{aligned} -G = {}&\epsilon_1\ln(\epsilon - \epsilon_1) - \frac{1}{B}\ln(1 - B\epsilon) + \epsilon_2\ln(\epsilon - \epsilon_2) - \frac{1}{B}\ln(1 - B\epsilon) \\ &-\left[B\ln(\epsilon - B) - \frac{1}{B}\ln(1 - B\epsilon)\right] \end{aligned} \tag{k}$$

Since

$$\frac{\pi U}{w_o} = \frac{i}{2}(F + G) + c_1 \tag{ℓ}$$

$$\frac{\pi V}{w_o} = \frac{1}{2}\left[-F + G\right] + c_2 \tag{m}$$

where c_1 and c_2 are constants of integration, then

$$-\frac{2\pi U}{iw_o} = -F - G + c_1 = \left(-\frac{1 - B\epsilon_1}{\epsilon_1} - \frac{1 - B\epsilon_2}{\epsilon_2} + \frac{1 - B^2}{B}\right)\ell n\,\epsilon$$

$$+ \left(\frac{1}{\epsilon_1} + \epsilon_1\right)\ell n(\epsilon - \epsilon_1) + \left(\frac{1}{\epsilon_2} + \epsilon_2\right)\ell n(\epsilon - \epsilon_2)$$

$$-\left(\frac{1}{B} + B\right)\ell n\,(1 - B\epsilon) - \left(\frac{1}{B} + B\right)\ell n(\epsilon - B) + c_1$$

$$= \left(\frac{1 + B^2}{B} - 2\,\mathrm{Re}\,\epsilon_1\right)\ell n\,\epsilon + 2\mathrm{Re}(\epsilon_1)\,\ell n(\epsilon - \epsilon_1)(\epsilon - \epsilon_2)$$

$$-\left(B + \frac{1}{B}\right)\ell n(\epsilon - B)(1 - B\epsilon) + c_1 \tag{n}$$

or

$$\frac{\pi U}{w_o} = i\left[\mathrm{Re}\,\epsilon_1 - \frac{1 + B^2}{2B}\right]\ell n\,\epsilon - i\mathrm{Re}(\epsilon_1)\,\ell n(\epsilon - \epsilon_1)(\epsilon - \epsilon_2)$$

$$+ \frac{i}{2}\left(B + \frac{1}{B}\right)\ell n(\epsilon - B)(1 - B\epsilon) + c_1 \tag{4.5}$$

$$\frac{2\pi V}{w_o} = -F + G = \left(-\frac{1 - B\epsilon_1}{\epsilon_1} - \frac{1 - B\epsilon_2}{\epsilon_2} + \frac{1 - B^2}{B}\right)\ell n\,\epsilon$$

$$+ (\epsilon_2 - \epsilon_1)\,\ell n(\epsilon - \epsilon_1) + (\epsilon_1 - \epsilon_2)\ell n(\epsilon - \epsilon_2)$$

$$+ \left(\frac{1}{B} - B\right)\ell n(1 - B\epsilon) + \left(-\frac{1}{B} + B\right)\ell n(\epsilon - B) + c_2 \tag{o}$$

or

$$\frac{\pi V}{w_o} = \left(\frac{1+B^2}{2B} - \mathrm{Re}\,\epsilon_1\right) \ln \epsilon + i(\mathrm{Im}\,\epsilon_1) \ln \frac{\epsilon - \epsilon_2}{\epsilon - \epsilon_1} + \tfrac{1}{2}\left(B - \frac{1}{B}\right) \ln \frac{\epsilon - B}{1 - B\epsilon} + c_2 \tag{4.6}$$

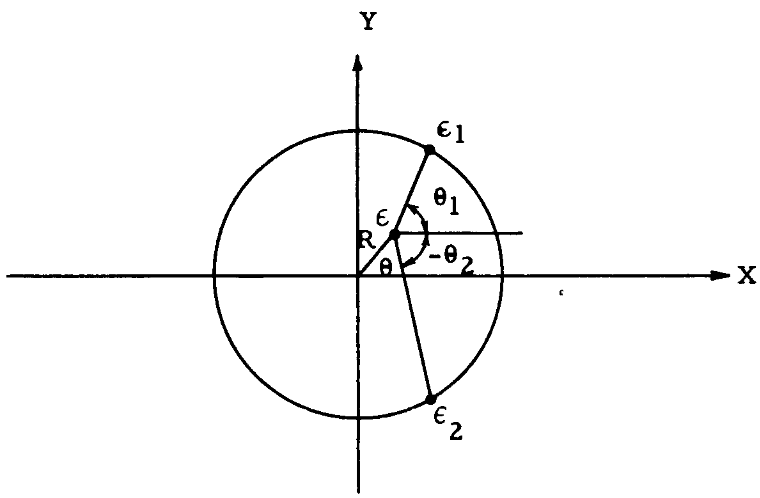


FIGURE 20. NOTATION USED IN EVALUATING EQUATION (4.6)

The most convenient way to evaluate the constants c_1 and c_2 from the boundary conditions is first to find the real parts of V and U. Figure 20 gives for a point $\epsilon = R \cdot e^{i\theta}$

$$\tan \theta_1 = \frac{\sin \psi - R \sin\theta}{\cos\psi - R \cos\theta}$$

$$-\tan \theta_2 = \frac{\sin \psi + R \sin\theta}{\cos\psi - R \cos\theta}$$

Hence

$$\tan\left\{\operatorname{Re}\left[-i\,\ell n(\epsilon-\epsilon_1)(\epsilon-\epsilon_2)\right]\right\} = \tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

$$= \frac{\dfrac{-2R\sin\theta}{\cos\psi - R\cos\theta}}{1+\dfrac{\sin^2\psi - R^2\sin^2\theta}{(\cos\psi - R\cos\theta)^2}}$$

$$= \frac{-2R\sin\theta\,(\cos\psi - R\cos\theta)}{1+R^2(\cos^2\theta-\sin^2\theta) - 2R\cos\theta\cos\psi} \qquad \text{(p)}$$

Similarly

$$\tan\left\{\operatorname{Re}\left[i\,\ell n\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1}\right]\right\} = \tan(\theta_1-\theta_2) = \frac{\dfrac{2\sin\psi}{\cos\psi - R\cos\theta}}{1+\dfrac{R^2\sin^2\theta-\sin^2\psi}{(\cos\psi - R\cos\theta)^2}}$$

$$= \frac{2\sin\psi(-\cos\psi + R\cos\theta)}{1-R^2-2\cos^2\psi + 2R\cos\psi\cos\theta} \qquad \text{(q)}$$

To evaluate c_1 and c_2, put $\epsilon = 1$; i.e., $R = 1$, $\cos\theta = 1$. All the terms in Equation (4.5) have vanishing real parts, i.e., $c_1 = 0$. All the terms in Equation (4.6) vanish except the term computed in formula (q) above. The right-hand side of formula (q) reduces to

$$\frac{2\sin\psi(-\cos\psi+1)}{-2\cos^2\psi+2\cos\psi} = \frac{\sin\psi}{\cos\psi} = \frac{\sqrt{1-\tan^2\phi}}{\tan\phi}$$

Hence

$$c_2 = -(\operatorname{Im}\epsilon_1)\left[\operatorname{Re}\left(i\,\ell n\frac{1-\epsilon_2}{1-\epsilon_1}\right)\right] = -\sqrt{1-\tan^2\phi}\ \arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi} \qquad \text{(r)}$$

Introducing these values and some standard simplifications gives

$$\frac{\pi U}{w_o} = i\left[\tan\phi - \mathrm{ctn}\beta\right]\ln\epsilon - i\tan\phi\ \ln(\epsilon-\epsilon_1)(\epsilon-\epsilon_2) + i\,\mathrm{ctn}\beta\ln(\epsilon - B)(1 - B\epsilon) \tag{4.7a}$$

and

$$\frac{\pi V}{w_o} = (\mathrm{ctn}\beta - \tan\phi)\ln\epsilon + i\sqrt{1-\tan^2\phi}\,\ln\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1} + \left(\frac{B^2-1}{2B}\right)\ln\frac{\epsilon - B}{1 - B\epsilon} - \sqrt{1-\tan^2\phi}\,\arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi} \tag{4.7b}$$

To complete the evaluation of the real parts the third term in Equation (4.7a) should also be considered. With reference to Figure 21

$$\mathrm{Re}\left[(i\ln(\epsilon - B)(1 - B\epsilon)\right] = -(\theta_1 + \theta_2) + n\pi$$

where

$$n = \pm 1 \text{ for } B > 0$$

$$= 0 \text{ for } B < 0$$

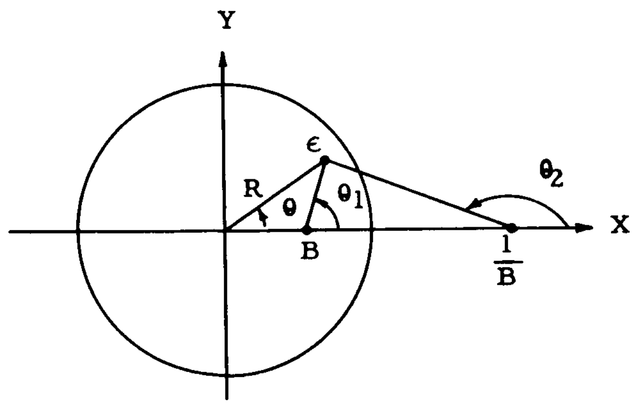


FIGURE 21. ϵ-PLANE

The angles θ_1 and θ_2 are determined by

$$\tan\theta_1 = \frac{R \sin\theta}{R\cos\theta - B}$$

$$\tan\theta_2 = \frac{R \sin\theta}{R\cos\theta - \frac{1}{B}} \qquad \text{(s)}$$

$$\tan(\theta_1 + \theta_2) = \frac{R\sin\theta\left[2R\cos\theta - (B + \frac{1}{B})\right]}{R^2(\cos^2\theta - \sin^2\theta) - R\cos\theta\,(B + \frac{1}{B}) + 1}$$

Hence

$$\frac{\pi u}{w_o} = (-\tan\phi + \frac{1}{b})\,\theta + \tan\phi \quad \arctan\frac{-2R\sin\theta\,(\tan\phi - R\cos\theta)}{1 - R^2(\sin^2\theta - \cos^2\theta) - 2R\cos\theta\tan\varphi}$$

$$-\frac{1}{b}\left(\arctan\frac{2R\sin\theta\,(R\cos\theta - \frac{1}{b})}{R^2(\cos^2\theta - \sin^2\theta) - 2\frac{R}{b}\cos\theta + 1} + n\pi\right) \qquad (4.8a)$$

where $n = 0$ for $b < 0$, $n = -1$ for $b > 0$. Actually $n\pi$ = - argument of $(-B)$. The values of arc tan and n have to be chosen so that $u = 0$ for $\epsilon = 1$.

To find the real part of $\ln\frac{\epsilon - B}{1 - B}$ in Equation (4.7b) it may be observed that if

$$|\epsilon - B| = R_1 \text{ and } |1 - B\epsilon| = R_2$$

then

$$R_1^2 = R^2 + B^2 - 2BR\cos\theta$$

and

$$R_2^2 = 1 + B^2R^2 - 2BR\cos\theta$$

Hence

$$\mathrm{Re}\left(\ln \frac{\epsilon - B}{1 - B\epsilon}\right) = \ln R_1 - \ln R_2 = \tfrac{1}{2} \ln \frac{R^2 + B^2 - 2BR\cos\theta}{1 + B^2R^2 - 2BR\cos\theta}$$

The expression for v then becomes

$$\frac{\pi v}{w_o} = (\operatorname{ctn}\beta - \tan\phi)\ln R + \left(\sqrt{1-\tan^2\phi}\right)\left[\arctan \frac{2\sqrt{1-\tan^2\phi}\ (R\cos\theta - \tan\phi)}{1-R^2-2\tan^2\phi + 2R\tan\phi\cos\theta}\right.$$

$$\left. - \arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi}\right] + \frac{B^2-1}{4B}\ln\frac{R^2 + B^2 - 2BR\cos\theta}{1 + B^2R^2 - 2BR\cos\theta} \qquad (4.8b)$$

Formulas (4.8a) and (4.8b) will now be specialized to points in the plane of the wing. For such points θ is either 0, + π (top side), or $-\pi$ (bottom side). This value has to be substituted in Equations (4.8a) and (4.8b), or Equation (4.7) may be used. Care has to be exercised in choosing the values of the arc tan function. A reference to Figures 20 and 21 is here recommended.

The following calculations apply to the top side of the plane of the wing. Because of antisymmetry, u takes on the corresponding values of opposite sign on the bottom side.

The expression for u becomes

$$\frac{\pi u}{w_o} = \left[-\tan\phi + \operatorname{ctn}\beta\right] \arg\epsilon - \operatorname{ctn}\beta\left[\arg(\epsilon - B) + \arg(1 - B\epsilon)\right]$$

$$\frac{u}{w_o} = (-\tan\phi + \operatorname{ctn}\beta)\, n_1 - n_2 \operatorname{ctn}\beta \qquad (4.9a)$$

where

$n_1 = 1$ for $t < 0$. $\qquad n_2 = 1$ for $t < b$

$= 0$ for $t > 0$. $\qquad = 0$ for $t > b$

The corresponding formula for v reads

$$\frac{\pi v}{w_o} = (\text{ctn}\,\beta - \tan\phi)\ln|T| + \left(\sqrt{1-\tan^2\phi}\right)\left[\arc\tan\frac{2\sqrt{1-\tan^2\phi}\,(T-\tan\phi)}{1-T^2-2\tan^2\phi+2T\tan\phi)}\right.$$

$$\left. - \arc\tan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi}\right] + \frac{B^2-1}{2B}\ln\left|\frac{B-T}{BT-1}\right| \tag{4.9b}$$

Planform II will now be specialized in four different ways by assuming simple values of the angles ϕ and β : (1) $\phi = 0$, (2) $\beta = -\pi/4$, (3) $\phi = 0$, $\beta = -\pi/4$, and (4) $\beta = 0$. In the first case ($\phi = 0$), Equations (4.7) and (4.8) reduce to

$$\frac{\pi U}{w_o} = -\frac{i}{b}\left[\ln\epsilon - \ln(\epsilon - B)(1 - B\epsilon)\right] \tag{4.10a}$$

$$\frac{\pi V}{w_o} = \frac{B^2+1}{2B}\ln\epsilon - i\ln\frac{\epsilon - i}{\epsilon + i} + \frac{B^2-1}{2B}\ln\frac{\epsilon - B}{1-B\epsilon} - \frac{\pi}{2} \tag{4.10b}$$

$$\frac{\pi u}{w_o} = -\text{ctn}\,\beta\left[-\theta + \arc\tan\left(\frac{2R\sin\theta\,(R\cos\theta - \text{ctn}\,\beta)}{R^2(\cos^2\theta - \sin^2\theta) - R\cos\theta\,\text{ctn}\,\beta + 1}\right) + n\pi\right] \tag{4.11a}$$

$$\frac{\pi v}{w_o} = \frac{1}{b}\ln R + \arc\tan\frac{2R\cos\theta}{1-R^2}$$

$$+ \frac{B^2-1}{4B}\ln\frac{R^2+B^2-2BR\cos\theta}{1+B^2R^2-2BR\cos\theta} - \frac{\pi}{2} \tag{4.11b}$$

For points in the plane of the wing, there is a further reduction:

$$\frac{u}{w_o} = (n_1 - n_2)\,\text{ctn}\,\beta$$

$$n_1 = 1 \text{ for } t < 0,\quad n_1 = 0 \text{ for } t > 0 \tag{4.12a}$$

$$n_2 = 1 \text{ for } t < b,\quad n_2 = 0 \text{ for } t > b$$

$$\frac{\pi v}{w_o} = \frac{1}{b}\ln|T| + \arc\tan\frac{2T}{1-T^2} + \frac{B^2-1}{2B}\ln\left|\frac{T-B}{1-BT}\right| - \frac{\pi}{2} \tag{4.12b}$$

In the second case $(\beta = -\pi/4)$ one obtains

$$\frac{\pi U}{w_o} = i\,(\tan\phi + 1)\ln\epsilon - i\tan\phi\ln(\epsilon-\epsilon_1)(\epsilon-\epsilon_2) - 2i\ln(1+\epsilon) \qquad (4.13a)$$

$$\frac{\pi V}{w_o} = -(1+\tan\phi)\ln\epsilon + i\sqrt{1-\tan^2\phi}\,\ln\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1}$$

$$-\sqrt{1-\tan^2\phi}\,\arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi} \qquad (4.13b)$$

$$\frac{\pi u}{w_o} = -(\tan\phi+1)\theta + \tan\phi\cdot\arctan\frac{-2R\sin\theta\,(\tan\phi - R\cos\theta)}{1+R^2(\cos^2\theta-\sin^2\theta)-2R\cos\theta\tan\phi}$$

$$+\,2\arctan\frac{R\sin\theta}{1+R\cos\theta} \qquad (4.14a)$$

$$\frac{\pi v}{w_o} = -(1+\tan\phi)\ln R + \sqrt{1-\tan^2\phi}\left[\arctan\frac{2\sqrt{1-\tan^2\phi}\,(R\cos\theta-\tan\phi)}{1-R^2-2\tan^2\phi+2R\tan\phi\cos\theta}\right.$$

$$\left.-\arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi}\right] \qquad (4.14b)$$

For points in the plane of the wing:

$$\frac{u}{w_o} = -(\tan\phi + 1)n_1 \qquad (4.15a)$$

$$n_1 = 1 \text{ for } t < 0, \quad n_1 = 0 \text{ for } t > 0$$

$$\frac{\pi v}{w_o} = -(1+\tan\phi)\ln|T|$$

$$+\sqrt{1-\tan^2\phi}\left[\arctan\frac{2\sqrt{1-\tan^2\phi}\,(T-\tan\phi)}{1-T^2-2\tan^2\phi+2T\tan\phi}\right.$$

$$\left.-\arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi}\right] \qquad (4.15b)$$

In the third case $(\phi = 0,\ \beta = -\pi/4)$ one obtains

$$\frac{\pi U}{w_o} = i\ell n\,\epsilon - 2\,i\ell\,n(1+\epsilon) \tag{4.16a}$$

$$\frac{\pi V}{w_o} = -\ell n\,\epsilon + i\ell n\frac{\epsilon+i}{\epsilon-i} - \frac{\pi}{2} \tag{4.16b}$$

$$\frac{\pi u}{w_o} = -\theta + 2\arctan\frac{R\sin\theta}{1+R\cos\theta} \tag{4.17a}$$

$$\frac{\pi v}{w_o} = -\ell nR + \arctan\frac{2R\cos\theta}{1-R^2} - \frac{\pi}{2} \tag{4.17b}$$

For points in the plane of the wing:

$$\frac{u}{w_o} = -1 \text{ for } t < 0, \quad u = 0 \text{ for } t > 0 \tag{4.18a}$$

$$\frac{\pi v}{w_o} = -\ell n\,|T| + \arctan\frac{2T}{1-T^2} - \frac{\pi}{2} \tag{4.18b}$$

In the fourth special case ($\beta = 0$) the following limits have to be evaluated:

$$\left.\begin{aligned}\lim_{B\to 0}\frac{\ell n(\epsilon-B)(1-B\epsilon)-\ell n\epsilon}{b} &= \lim_{B\to 0}\frac{\ell n(\epsilon-B)(1-B\epsilon)-\ell n\epsilon}{2B}\\ &= \frac{1}{2}\left[\frac{d}{dB}\,\ell n(\epsilon-B)(1-B\epsilon)\right]_{B=0}\\ &= -\frac{1}{2}\left(\epsilon+\frac{1}{\epsilon}\right)\end{aligned}\right\} \quad \text{(a)}$$

$$\lim_{B\to 0} -\frac{1}{2B}\left(\ell n\frac{\epsilon-B}{1-B\epsilon} - \ell n\,\epsilon\right) = -\frac{1}{2}\left[\frac{d}{dB}\,\ell n\,\frac{\epsilon-B}{1-B\epsilon}\right]_{B=0} = \frac{1}{2}\left(\frac{1}{\epsilon}-\epsilon\right) \quad \text{(b)}$$

Then Equations (4.7a) and (4.7b) reduce to

$$\frac{\pi U}{w_o} = i\tan\phi\,\ell n\,\epsilon - i\tan\phi\,\ell n(\epsilon-\epsilon_1)(\epsilon-\epsilon_2) - \frac{i}{2}\left(\epsilon+\frac{1}{\epsilon}\right) \tag{4.19a}$$

$$\frac{\pi V}{w_o} = -\tan\phi \,\ln\epsilon + i\sqrt{1-\tan^2\phi}\,\ln\frac{\epsilon-\epsilon_2}{\epsilon-\epsilon_1} + \tfrac{1}{2}\left(\frac{1}{\epsilon}-\epsilon\right)$$
$$-\sqrt{1-\tan^2\phi}\ \arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi} \qquad (4.19b)$$

The corresponding real parts are most easily found by comparing Equations (4.19a) and (4.19b) with Equations '4.7a), (4.7b), (4.8a), and (4.8b).

$$\frac{\pi u}{w_o} = -\theta\tan\phi + \tan\phi\ \arctan\frac{-2R\sin\theta(\tan\phi - R\cos\theta)}{1-R^2(\sin^2\theta-\cos^2\theta)-2R\tan\phi\cos\theta}$$
$$+\frac{\sin\theta}{2}\left(R-\frac{1}{R}\right) \qquad (4.20a)$$

$$\frac{\pi v}{w_o} = -\tan\phi\,\ln R + \sqrt{1-\tan^2\phi}\left[\arctan\frac{2\sqrt{1-\tan^2\phi}\,(R\cos\theta-\tan\phi)}{1-R^2-2\tan^2\phi+2R\tan\phi\cos\theta}\right.$$
$$\left.-\arctan\frac{\sqrt{1-\tan^2\phi}}{\tan\phi}\right] + \frac{\cos\theta}{2}\left(\frac{1}{R}-R\right) \qquad (4.20b)$$

B. Note on Applications

Probably the most important application of the wings of constant-lift distribution is in finding the induced flow field behind a wing of finite chord. This application was indicated by Busemann (Cf. Ref. 4) and is discussed in detail in Reference 2.

V. MIXED TYPE OF LIFTING WING

The problems studied in this section will be those in which the lift distribution is prescribed over one region of the wing and in which the shape is prescribed for the rest of the wing. Only a special case will be treated with the following properties: In one region the lift distribution is constant, $w = \pm w_o$, and the rest of the wing is a flat plate at zero angle of attack. The importance of such solutions is that the interference of a lifting element is determined (Cf. Section V-D). In Section IV wings of constant-lift distribution were discussed. In the plane of the wing such wings induce a downwash field but no pressure field off the wing. However, if a flat plate at zero angle of attack is introduced into this downwash field, an interference effect results. Since the downwash has to be zero at the flat plate, an additional flow field is generated which cancels the downwash due to the lifting element at the plate, and at the same time yields lift on the flat plate. Thus, though actually the flat plate is at zero angle of attack, it has a variable effective angle of attack because of the downwash field. The resulting composite flow field will be determined in this section, and the lift induced by a flat lifting element will be discussed in Section VI.

Solutions in this section will be straightforward, because of the previously developed technique, in particular the use of the flatness condition (1.46) in combination with analytic extension by reflection and also the use of the Lorentz transformation.

The previous classification of wings by planforms is no longer sufficient. Only certain representative examples of the various possible cases will be treated.

A. Example I: Pressure Distribution

The most important case is probably the one shown in Figure 22. This wing consists of two regions. Region I is flat and at zero angle of attack ($v = 0$), and region II has a constant-lift distribution $w = \pm w_o$. The more

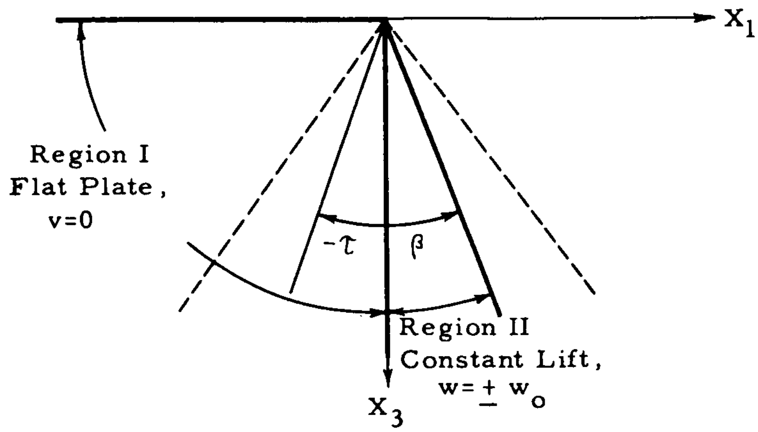


FIGURE 22. MIXED TYPE OF WING, EXAMPLE I

general case $\alpha \neq 0$ on region I can be obtained easily by simply adding a solution from Section III which has the same planform as region I and which has a constant angle of attack $\neq 0$. Such a wing would have $w = 0$ on region II and hence its superposition would not spoil the boundary condition: $w = \pm w_o$ on region II. It is also evident that when the flat plate (region I) is at zero angle of attack, the sweepback angle of the leading edge of region I is irrelevant as long as this edge is outside the Mach cone; in this case any perturbation velocity existing in region I is induced by region II and there are no disturbances outside the Mach cone.

The boundary conditions in the ϵ-plane are shown in Figure 23.

The fact that $\alpha = 0$ on region I and on the wing outside the Mach cone has as a consequence that $w = 0$ on the Mach circle. Furthermore, since v

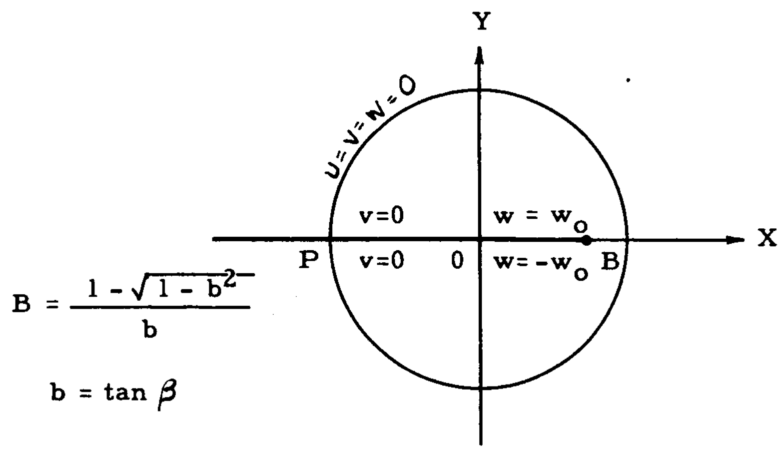


FIGURE 23. BOUNDARY CONDITIONS IN THE ϵ-PLANE, EXAMPLE I

is constant on line OP (Cf. Fig. 23), it is assumed as before that $w^* = 0$ here by Equation (1.46'). As in Section III the function W is extended by reflection in OP. Assume, to begin with, that there is a function W which is analytic inside the unit circle except for the slits OP and OB. By reflection in OP, W is extended to a two-sheeted surface consisting of two unit circles. The point 0 is then a branch point, and passage from one sheet to the other is across the line OP. On the lower sheet a slit OB is then obtained similar to the one on the upper surface. As before, the surface is made single sheeted by the transformation

$$\nu = \sqrt{\epsilon} \tag{5.1}$$

The boundary conditions in the ν-plane are shown in Figure 24.

This is a simple type of boundary value problem. The narrow unyawed

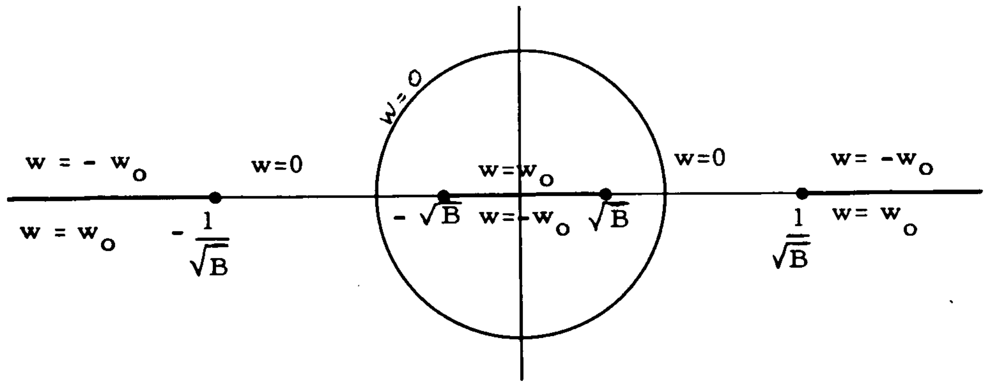


FIGURE 24. BOUNDARY CONDITIONS IN THE ν-PLANE, EXAMPLE I

delta wing (planform III) has these boundary values for w in the ϵ -plane if it has constant lift (Cf. Section IV) and for v in the ϵ -plane if it is nonlifting and has a symmetrical wedge profile (Cf. Section II). There are two logarithmic singularities at $\pm\sqrt{B}$. Since w = 0 on Mach circle, two additional singularities are obtained at $\pm 1/\sqrt{B}$ by reflection. By the standard procedure previously used in this report, the strength of the logarithmic singularities can be determined, and when the singularities are known, the function may be written down immediately

$$W = \frac{-i\, w_o}{\pi} \ln \frac{(\nu - \sqrt{B})(1 - \sqrt{B}\,\nu)}{(\nu + \sqrt{B})(1 + \sqrt{B}\,\nu)} \tag{5.2}$$

This discussion of the singularities does not determine whether a constant is to be added to the right-hand side of Equation (5.2). That Equation (5.2) actually is the correct expression if suitable values of the multivalued logarithm function are taken will be seen from the evaluation of w in terms of variables in physical space (Cf. proof of Eq. 5.3).

For the applications, one is primarily interested in the lift distribution in region I. Since it is at zero angle of attack, it would have zero lift by itself. However, the presence of the lifting surface, region II, induces a lift in region I. This region corresponds to the negative real axis in the ϵ -plane and to the imaginary axis in the ν -plane. A ray in region I is characterized by the angle τ , which is negative.

$$t = \tan \tau, \qquad T = \frac{1 - \sqrt{1 - t^2}}{t} \tag{a}$$

and also

$$N = \sqrt{-T} \tag{b}$$

The ray τ then corresponds to the point Ni (or -Ni) in the ν -plane.

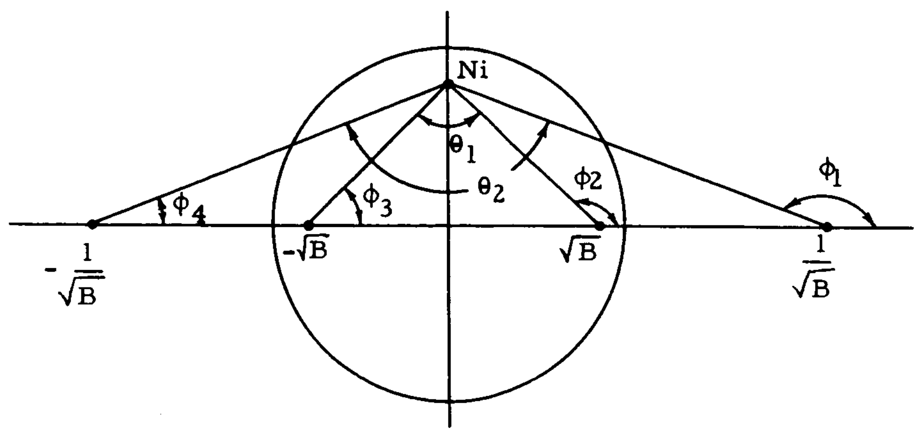


FIGURE 25. ν -PLANE: EXAMPLE I

From Equation (5.2) it follows that the value of w at the point Ni is given by

$$\frac{\pi w}{w_o} = \phi_1 + \phi_2 - \phi_3 - \phi_4 - \pi = \theta_1 + \theta_2 - \pi \qquad \text{(c)}$$

where the notation is as explained in Figure 25. Equation (c) is still true if the point P is on the unit circle, in which case it gives $w = 0$, the correct boundary condition. It also gives the correct value $w = w_o$ on interval $(-\sqrt{B}, \sqrt{B})$ which corresponds to region II on the wing.

From Equation (c) one may evaluate w as a function of τ on region I.

$$\tan\frac{\theta_1}{2} = \frac{\sqrt{B}}{N}, \qquad \tan\frac{\theta_2}{2} = \frac{1}{\sqrt{B}\,N} \qquad \text{(d)}$$

$$\tan\frac{\theta_1+\theta_2}{2} = \frac{\frac{\sqrt{B}}{N} + \frac{1}{N\sqrt{B}}}{1 - \frac{1}{N^2}} = \frac{N(B+1)}{\sqrt{B}\,(N^2-1)} \qquad \text{(e)}$$

$$\cos(\theta_1+\theta_2) = \frac{1 - \frac{N^2(B+1)^2}{(N^2-1)^2\,B}}{1 + \frac{N^2(B+1)^2}{(N^2-1)^2 B}} \qquad \text{(f)}$$

But by Equation (1.37c)

$$\frac{(B+1)^2}{B} = \left(\frac{B^2+1}{2B}\right)\cdot 2 + 2 = 2\left(\frac{1}{b}+1\right) = 2\left(\frac{b+1}{b}\right) \qquad \text{(g)}$$

and

$$\frac{(N^2-1)^2}{N^2} = \frac{(-T-1)^2}{-T} = \left(\frac{T^2+1}{-2T}\right)\cdot 2 - 2 = 2\,\frac{-1-t}{t} \qquad \text{(h)}$$

Introducing Equations (g) and (h) into Equation (f):

$$\cos(\theta_1 + \theta_2) = \frac{1 + \frac{t}{1+t}\,\frac{b+1}{b}}{1 - \frac{t}{1+t}\,\frac{b+1}{b}} = \frac{b + t + 2bt}{b - t} \qquad \text{(i)}$$

Using the formula $(\arc\cos x) - \pi = \arc\cos(-x)$, Equations (c) and (i) result in

$$\frac{w}{w_o} = \frac{1}{\pi}\,\text{arc cos}\,\frac{b + t + 2bt}{t - b}$$

$$t = \tan\tau \qquad (5.3)$$

For the top side of region I, arc cos is to be taken between 0 and $+\pi$. The boundary between regions I and II corresponds to $t = 0$. Here $w = (w_o/\pi)$ arc cos $(b/-b) = w_o$, which checks. Where the Mach cone cuts region I, $t = -1$. Then $w/w_o = \frac{1}{\pi}$ arc cos $(b - 1 - 2b) / (-1 - b) = 0$. This value also checks since region II cannot induce any lift outside the Mach cone.

B. Example I; Sidewash and Downwash

U and V can be calculated most readily following the procedure in Section III-E. Introducing $\nu = \sqrt{\epsilon}$ in Equations (1.42)

$$\frac{dU}{d\nu} = -\tfrac{1}{2}\left(\nu^2 + \frac{1}{\nu^2}\right)\frac{dW}{d\nu} \qquad \text{(a)}$$

$$\frac{dV}{d\nu} = \frac{i}{2}\left(\nu^2 - \frac{1}{\nu^2}\right)\frac{dW}{d\nu} \qquad \text{(b)}$$

Let

$$\frac{dF}{d\nu} = -\frac{\pi}{iw_o}\,\frac{1}{\nu^2}\,\frac{dW}{d\nu} \qquad \text{(c)}$$

$$\frac{dG}{d\nu} = -\frac{\pi}{iw_o}\cdot\nu^2\,\frac{dW}{d\nu} \qquad \text{(d)}$$

Differentiating Equation (5.2)

$$\frac{dW}{d\nu} = -\frac{iw_o}{\pi}\left[\frac{1}{\nu - \sqrt{B}} + \frac{1}{\nu - \frac{1}{\sqrt{B}}} - \frac{1}{\nu + \sqrt{B}} - \frac{1}{\nu + \frac{1}{\sqrt{B}}}\right] \quad \text{(e)}$$

Expanding $\frac{1}{\nu^2}\frac{dW}{d\nu}$ and $\nu^2\frac{dW}{d\nu}$ into partial fractions

$$\frac{dF}{d\nu} = -2\left(\frac{B+1}{\sqrt{B}}\right)\frac{1}{\nu^2} + \left[\frac{1}{B(\nu - \sqrt{B})} + \frac{B}{\nu - \frac{1}{\sqrt{B}}} - \frac{1}{B(\nu + \sqrt{B})} - \frac{B}{\nu + \frac{1}{\sqrt{B}}}\right] \quad \text{(f)}$$

$$\frac{dG}{d\nu} = 2\left(\frac{B+1}{\sqrt{B}}\right) + \left[\frac{B}{\nu - \sqrt{B}} + \frac{1}{B\left(\nu - \frac{1}{\sqrt{B}}\right)} - \frac{B}{\nu + \sqrt{B}} - \frac{1}{B\left(\nu + \frac{1}{\sqrt{B}}\right)}\right] \quad \text{(g)}$$

Integrating Equations (f) and (g)

$$F = 2\left(\frac{B+1}{\sqrt{B}}\right)\frac{1}{\nu} + \left[\frac{1}{B}\ln(\nu - \sqrt{B}) + B\ln(1 - \nu\sqrt{B}) - \frac{1}{B}\ln(\nu + \sqrt{B}) - B\ln(1 + \nu\sqrt{B})\right]$$

$$+ \text{ constant} \quad \text{(h)}$$

$$G = 2\left(\frac{B+1}{\sqrt{B}}\right)\nu + \left[B\ln(\nu - \sqrt{B}) + \frac{1}{B}\ln(1 - \nu\sqrt{B}) - B\ln(\nu + \sqrt{B}) - \frac{1}{B}\ln(1 + \nu\sqrt{B})\right]$$

$$+ \text{ constant} \quad \text{(i)}$$

Then the complex sidewash and downwash are

$$\pi\frac{U}{w_o} = \frac{i}{2}(F + G) + \text{constant} \quad \text{(j)}$$

$$\pi\frac{V}{w_o} = \tfrac{1}{2}(-F + G) + \text{constant} \quad \text{(k)}$$

Hence

$$\pi\frac{U}{w_o} = i\left(\frac{B+1}{\sqrt{B}}\right)\left(\nu + \frac{1}{\nu}\right) + i\left(\frac{B^2+1}{2B}\right)\ln\frac{(\nu - \sqrt{B})(1 - \nu\sqrt{B})}{(\nu + \sqrt{B})(1 + \nu\sqrt{B})} + \text{constant} \quad (5.4)$$

$$\pi\frac{V}{w_o} = \left(\frac{B+1}{\sqrt{B}}\right)\left(\nu - \frac{1}{\nu}\right) - \left(\frac{1 - B^2}{2B}\right)\ln\frac{(\nu - \sqrt{B})(1 + \nu\sqrt{B})}{(\nu + \sqrt{B})(1 - \nu\sqrt{B})} + \text{constant} \quad (5.5)$$

The evaluation of the real sidewash and downwash (Cf. Eqs. 5.6 and 5.7) shows that both constants of integration in Equations (5.4) and (5.5) must be zero in order to satisfy the boundary conditions.

Of primary interest is the sidewash in the plane of the wing. Since the second term in Equation (5.4) is $-\left(\frac{B^2+1}{2B}\right)\left(\pi\frac{W}{w_o}\right)$, the sidewash is

$$\pi\frac{u}{w_o} = \left(\frac{B+1}{\sqrt{B}}\right)\left(\frac{1-R}{\sqrt{R}}\right)\sin\frac{\theta}{2} - \left(\frac{B^2+1}{2B}\right)\operatorname{Re}\left(\pi\frac{W}{w_o}\right) + c_1 \qquad (5.4')$$

The constant $c_1 = 0$ in order to satisfy the condition $U = 0$ at $\epsilon = -1$ (or $R = 1$, $\theta = \pm\pi$):

Transforming Equation (5.4') to physical coordinates gives, with the aid of Equations (1.36), (1.37c), and (1.37i),

$$\frac{\pi u}{w_o} = \sqrt{\frac{2(1+b)}{b}}\sqrt{\frac{2(1-|t|)}{|t|}}\sin\frac{\theta}{2} - \frac{1}{b}\operatorname{Re}\left(\frac{\pi W}{w_o}\right) \qquad (\ell)$$

For the sidewash in the plane of the wing three different cases have to be considered, namely, $(-1 \leq t < 0)$, $(0 < t < b)$, and $(b < t \leq 1)$.

In Region I, $\sin\theta/2$ is $= 1$, $-1 \leq t < 0$, and hence $|t| = -t$. The last term in Equation (ℓ) is given by Equation (5.3). Hence the sidewash is

$$\frac{\pi u}{w_o} = 2\sqrt{\frac{(1+b)(1+t)}{(-bt)}} - \frac{1}{b}\arccos\left(\frac{b+t+2bt}{t-b}\right) \qquad (5.6a)$$

for

$(-1 \leq t < 0)$.

In region II, $\sin\theta/2 = 0$, the first term of Equation (5.4) vanishes and $\operatorname{Re}(\pi W/w_o) = \pi$. Hence the sidewash is constant.

$$\frac{\pi u}{w_o} = -\frac{\pi}{b} \tag{5.6b}$$

for

$$(0 < t < b)$$

Finally for $b < t \leq 1$ it follows from Section I-J that $u = 0$.

Taking the real part of Equation (5.5), the downwash is found to be

$$\frac{\pi v}{w_o} = -\left(\frac{B+1}{\sqrt{B}}\right)\left(\frac{1-R}{\sqrt{R}}\right)\cos\frac{\theta}{2} - \left(\frac{1-B^2}{2B}\right)\ln\left|\frac{(\nu-\sqrt{B})(1+\nu\sqrt{B})}{(\nu+\sqrt{B})(1-\nu\sqrt{B})}\right| + c_2 \tag{m}$$

The constant $c_2 = 0$ in order to satisfy the condition $v = 0$ at $\epsilon = -1$ $(R = 1,\ \theta = \pm\pi,\ \nu = \pm i)$.

The downwash can now be obtained in the plane of the wing: In region I, $\theta = \pm\pi$ and ν is imaginary. Thus $v = 0$, which checks the boundary conditions. In region II, $\theta = 0$ and $\nu = \sqrt{T}$. Then

$$\left|\frac{(\nu-\sqrt{B})(1+\nu\sqrt{B})}{(\nu+\sqrt{B})(1-\nu\sqrt{B})}\right| = \left|\frac{(\sqrt{T}-\sqrt{B})(1+\sqrt{BT})}{(\sqrt{T}+\sqrt{B})(1-\sqrt{BT})}\right|$$

$$= \left|\frac{\frac{T^2+1}{2T} + \frac{B^2+1}{2B} + \frac{T-1}{\sqrt{T}}\cdot\frac{1-B}{\sqrt{B}} - 2}{\frac{T^2+1}{2T} - \frac{B^2+1}{2B}}\right|$$

Then with the aid of the formulas for the Tschaplygin transformation

$$\left|\frac{(\nu-\sqrt{B})(1+\nu\sqrt{B})}{(\nu+\sqrt{B})(1-\nu\sqrt{B})}\right| = \left|\frac{b + t - 2\sqrt{t(1-t)b(1-b)} - 2bt}{b-t}\right| \tag{n}$$

Hence the downwash in region II is

$$\frac{\pi v}{w_o} = -2\sqrt{\frac{(1+b)(1-t)}{bt}} - \frac{\sqrt{1-b^2}}{b}\ln\left|\frac{b + t - 2bt - 2\sqrt{bt(1-t)(1-b)}}{b-t}\right| \tag{5.7}$$

C. Example II

This case is obtained by yawing the wing in Example I. The boundary conditions on w and w* are given in Figure 26.

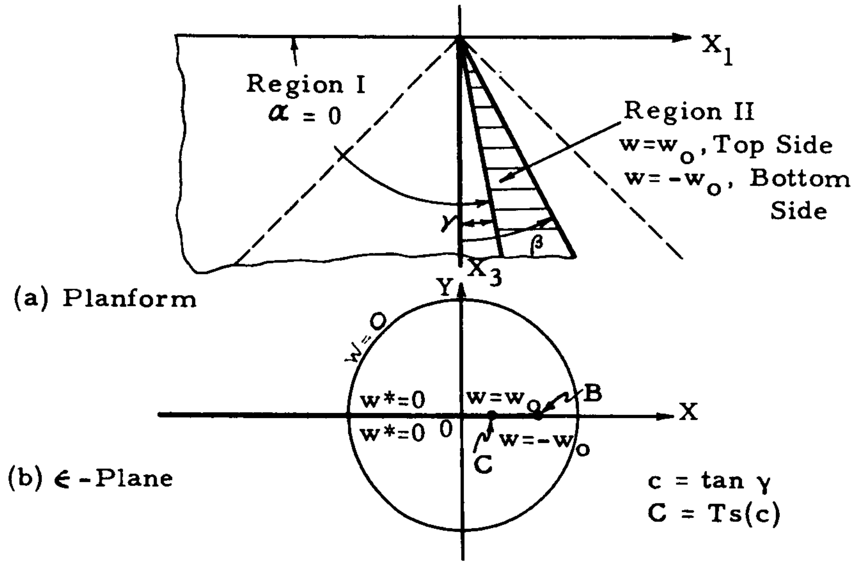


FIGURE 26. MIXED TYPE OF LIFTING WING, EXAMPLE II

The solution will of course be obtained from that for Example I by a Lorentz transformation. The situation is very similar to that in Section III (planform II). The transformation of the velocity potential may be used. The functions U, V, and W for Example II are then obtained immediately from Equations (1.57), (5.2), (5.4), and (5.5) and the values of u, v, and w in the plane of the wing from Equations (1.21c), (5.3), (5.6), and (5.7). However, the constant w_o has to be adjusted. If $w_o^{\,o}$ is the constant value of w on top of region II in Example I and $w_o^{\,c}$ the corresponding value for Example II, the formulas (1.21c) give the following relation between these two values:

$$w_o^c = \frac{1}{\sqrt{1-c^2}}\left[-c\left(\frac{-w_o^o}{b^*}\right)+w_o^o\right] = \frac{b\sqrt{1-c^2}}{b-c}\cdot w_o^o \qquad (5.8)$$

$$c = \tan\gamma, \qquad b^* = \frac{b-c}{1-bc}$$

This equation shows that w_o^c actually is a constant and also that w_o in the formulas (5.2), (5.4), and (5.5) has to be taken equal to $\frac{b-c}{b\sqrt{1-c^2}}\ w_o^c$ in order that the method of Lorentz transformation of the velocity potential give the prescribed value w_o^c in region II of the yawed wing.

The details of this transformation may be easily carried through as in Section III. However, only the other possibility will be worked out here, namely, the oblique transformation of the pressure function.

The solution will be written $W = W_1 + W_2$ where W_1 is given by Equation (5.2) after unyawing the wing by a homographic transformation

$$W_1 = \frac{-iw_o}{\pi}\, \ln\frac{(\nu-\sqrt{B^*})(1-\nu\sqrt{B^*})}{(\nu+\sqrt{B^*})(1+\nu\sqrt{B^*})} \qquad (5.9a)$$

where

$$\nu = \sqrt{\eta} = \sqrt{\frac{\epsilon - C}{1-C\epsilon}}\ ,\ B^* = \frac{B-C}{1-CB},\ C = Ts(c)$$

and the correction function W_2 is 0 for $\gamma < 0$ and is of the same form as Equation (3.30) for $\gamma > 0$

$$W_2 = \frac{ikw_o}{\pi}\left(\nu + \frac{1}{\nu}\right) \qquad (5.9b)$$

where k is a real number and must be evaluated to satisfy the condition

$$\left.\frac{d(W_1 + W_2)}{d\epsilon}\right|_{\epsilon = 0} = 0 \tag{5.10}$$

That this is the right correction function is shown as in Section III. Note that $Re(W_2) = 0$ for ν real and positive. Hence the addition of W_2 does not violate the boundary condition that $w = w_o$ = constant in region II. The evaluation of k follows:

$$\left.\frac{i\pi}{w_o}\frac{dW_1}{d\nu}\right|_{\epsilon = 0} = -2\sqrt{B^*}\left[\frac{1}{C + B^*} + \frac{1}{1 + CB^*}\right]$$

$$= -\frac{2(1 + B)\sqrt{(B - C)(1 - CB)}}{B(1 - C^2)}$$

$$\left.\frac{i\pi}{w_o}\frac{dW_2}{d\nu}\right|_{\epsilon = 0} = -k(1 + \frac{1}{C})$$

Then

$$\left.\frac{d(W_1 + W_2)}{d\nu}\right|_{\epsilon=0} = \frac{iw_o}{\pi}\left[\frac{2(1 + B)\sqrt{(B - C)(1 - CB)}}{B(1 - C^2)} + \frac{k(1 + C)}{C}\right] = 0$$

and

$$k = \frac{-2C(1 + B)\sqrt{(B - C)(1 - CB)}}{B(1 - C^2)(1 + C)} \tag{5.11}$$

With the aid of Equation (1.37) this formula is easily transformed into

$$k = -\frac{c}{b(1 + c)}\sqrt{\frac{2(1 + b)(b - c)}{1 - c}} \tag{5.11'}$$

Finally the pressure function is

$$\frac{\pi W}{w_o} = \frac{\pi}{w_o}(W_1+W_2) = -i\ln\frac{(\nu-\sqrt{B^*})(1-\nu\sqrt{B^*})}{(\nu+\sqrt{B^*})(1+\nu\sqrt{B^*})}$$

$$-\frac{2iC(1+B)\sqrt{(B-C)(1-CB)}}{B(1-C^2)(1+C)}\left(\nu+\frac{1}{\nu}\right) \tag{5.12}$$

where

$$\nu = \sqrt{\frac{\epsilon - C}{1 - C\epsilon}}$$

The evaluation of $w = w_1 + w_2$ in the plane of the wing in terms of the physical coordinates follows:

$$\frac{\pi w_2}{w_o} = \mathrm{Re}\left[ik\left(\nu+\frac{1}{\nu}\right)\right] = k\left[\frac{1}{\sqrt{R^*}}-\sqrt{R^*}\right]\sin\frac{\theta^*}{2} \text{ for } \theta^* = 0, \pm\pi \tag{5.13}$$

$$= \pm k\left[\frac{1}{\sqrt{|T^*|}}-\sqrt{|T^*|}\right] \text{ for } \theta^* = \pm\pi$$

$$= 0 \qquad \text{for } \theta^* = 0$$

In region I in the plane of the wing with the aid of Equation (1.37i)

$$\frac{\pi w_2}{w_o} = k\left(\frac{1+T^*}{\sqrt{-T^*}}\right) = k\sqrt{\frac{2(1+t^*)}{(-t^*)}} \tag{5.14}$$

for $-1 \leq t^* \leq 0$

$-1 \leq t \leq c$

where

$$t^* = \frac{t-c}{1-ct}$$

The expression for w_1 is obtained from Equation (5.3), replacing b by b* and t by t*. Hence, the pressure in region I is given by the formula

$$\frac{\pi w}{w_o} = \text{arc cos}\, \frac{b^* + t^* + 2b^*t^*}{t^* - b^*} + k\sqrt{\frac{2(1 + t^*)}{(-t^*)}} \tag{5.15}$$

for $-1 \leq t^* \leq 0$

$-1 \leq t \leq c$

where

$$t^* = \frac{t - c}{1 - ct}, \qquad b^* = \frac{b - c}{1 - cb}$$

and $k = 0$ for $\gamma < 0$ and k is given by Equation (5.11) or (5.11') for $\gamma > 0$.

Inserting the values for t^*, b^*, and k, one may rewrite Equation (5.15) as

$$\frac{\pi w}{w_o} = \text{arc cos}\, \frac{(b - c)(1 + t) + (t - c)(1 + b)}{(t - b)(1 + c)} \quad \text{for } c < 0 \tag{5.15'a}$$

$$\frac{\pi w}{w_o} = \text{arc cos}\, \frac{(b - c)(1 + t) + (t - c)(1 + b)}{(t - b)(1 + c)} - \frac{2c}{b(1 + c)}\sqrt{\frac{(b - c)(1 + b)(1 + t)}{c - t}} \quad \text{for } c > 0 \tag{5.15'b}$$

These formulas are valid on the flat part of the wing, i.e., $-1 \leq t \leq c$.

D . Note on Applications

The use of the mixed type of wing for obtaining solutions is illustrated in Figure 27.

The solution for the flat lifting delta wing A_1BA_2 is given in Section III-J. If one wishes to have the solution for the corresponding wing with the tips $A_1C_1D_1$ and $A_2C_2D_2$ cut off, one has to take the basic solution for A_1BA_2 and then cancel the lift in the cut-off region by infinitely many mixed-type wings. A typical wing of this kind would have (a) constant

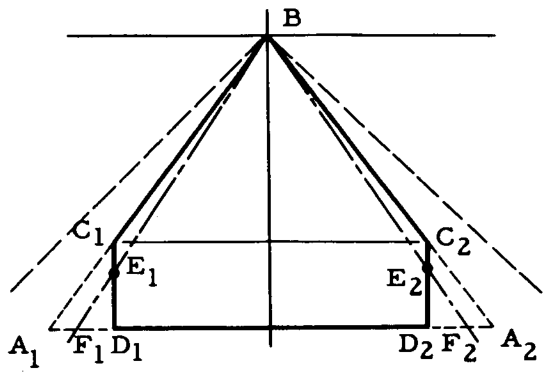


FIGURE 27. MODIFIED NARROW DELTA WING

lift on region $D_2E_2F_2$ (region II in Fig. 22) and (b) zero downwash to the left of this region (region I in Fig. 22). Condition (b) is necessary because the basic solution for A_1BA_2 already has the correct downwash on the wing $D_1C_1BC_2D_2$, and the superposition of additional solutions should not lead to a violation of the boundary condition that the wing is flat.

Another application is to wings of very low aspect ratio. The simplest example is the rectangular wing. If the aspect ratio is greater than unity, one obtains the solution by superposition of the solutions given in Section II-E and a two dimensional flow. However, if the aspect ratio is smaller than unity, one also needs solutions of the type studied in Section V-A (Cf. pp. 42 and 43 of Ref. 2).

Many similar applications are possible (Cf. also Section VI-E).

VI. WINGS WITH DISCONTINUOUS SLOPE

This section discusses conical wings with prescribed shape which are piecewise flat; that is, the surface of the wing consists of conical regions where α = constant and the value of the constant is different in adjacent regions. Thus α is discontinuous on the boundary between such regions.

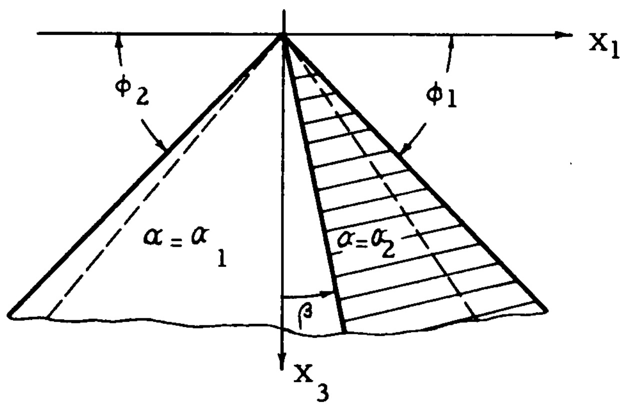
The symmetrical case is solved by trivial superposition of the solutions given in Section II. This is the same superposition principle that was used in Section II. As was explained there, it is justified because there is no downwash interference between symmetrical wings in the same plane.

The lifting case is also trivial when the upper and lower surfaces are independent, i.e., when all free edges are supersonic. In this case it is reduced to the symmetrical case as explained in Section III-A. This method is illustrated in Examples I and II of this section.

The case where at least one edge is subsonic is less easy. Examples III, IV, and V belong to this type. Although these examples may not be obtained from any symmetrical case, it will be seen that some results previously obtained in Section III will furnish part of the solution.

It will always be assumed that there is no gap at the place of discontinuity of angle of attack. By gap is meant a break in the solid surface through which the region above and below the wing are connected. In such a gap the pressure would have to be zero in the plane of the wing. If the gap is infinitely narrow, however, there is only a removable singularity; i.e., the pressure would have to be zero along the ray representing the gap, but this value would not influence the rest of the solution. It must be remembered, though, that a gap may have finite extension in the vertical direction and still its projection on the plane of the wing have zero width. This condition is easily visualized for a deflected control surface (Cf. Section VI-E). However, such a finite gap cannot be treated by the present theory which applies all boundary conditions on the wing to the corresponding projections in the plane of the wing.

Although there is a great variety of wings falling under the heading of this section, only a few examples are treated. They are in general of direct or indirect practical importance and are also chosen so as to illustrate the methods.



Profile: Flat Plate or Symmetrical Wedge

FIGURE 28. PLANFORM FOR EXAMPLE I

A. Example I

This wing is illustrated in Figure 28. It may be reduced to a simpler case. First of all, it is enough to consider the case of a symmetrical wedge profile. Then angle of attack means half angle of the wedge. From this case one obtains the solution for the flat plate profile by the method discussed at the beginning of Section II. Furthermore, it is sufficient to assume that the angle of attack = $(\alpha_1 - \alpha_2) = \lambda$ to the left of the line of discontinuity and = 0 to the right of this line. Then the general case may be obtained by superimposing a wing of the same planform and constant slope, $-\alpha_2$ (Cf. Section III-B).

Hence the problem has been reduced to the wing shown in Figure 29, but here the flat plate is immaterial. Thus there is simply the wing of

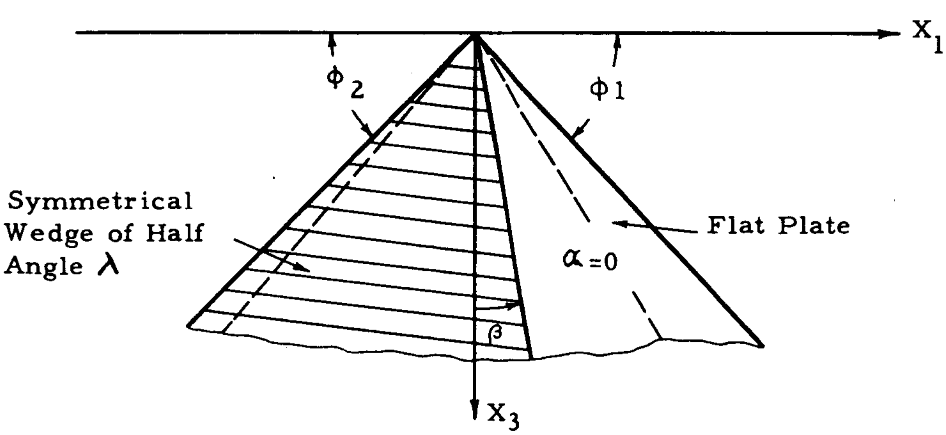
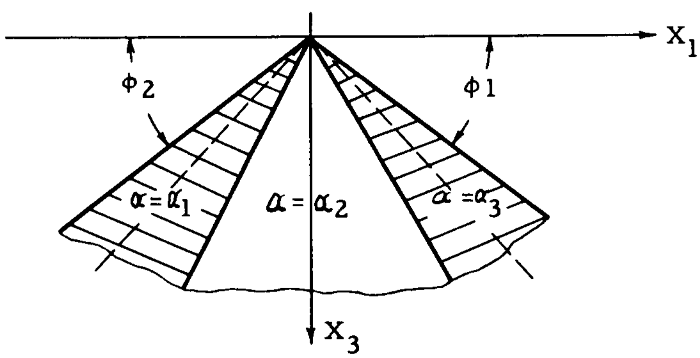


FIGURE 29. EXAMPLE I, REDUCED

planform II whose solution was given in Section II-A, for such a wing does not cause any downwash in the plane of the wing outside of the wing, as has been pointed out repeatedly. Hence a flat plate may always be inserted there without changing flow conditions.

Thus the pressure distribution on the wing in Figure 29 is given by formula (2.8c) for $\beta = 0$, and formula (2.8c) combined with formula (2.13a) for $\beta \neq 0$.

The principle used for treating Example I is evidently quite general. One may superimpose several nonlifting wings to form a nonlifting wing with prescribed slope distribution. If necessary, flat plates are then inserted to prevent any communication between the upper and lower surfaces. After they are inserted, one modifies the wing so that the lower surface is intact but the profile changes to a flat plate profile. Then one has a lifting wing whose angle of attack at any point is equal to half the wedge angle of the unmodified wing. The flow below the wing is the same in both cases, and the conditions above the wing are obtained by the general summetry principles of Section I. One more example of this principle is illustrated in Figure 30.



Profile: Symmetrical Wedge or Flat Plate

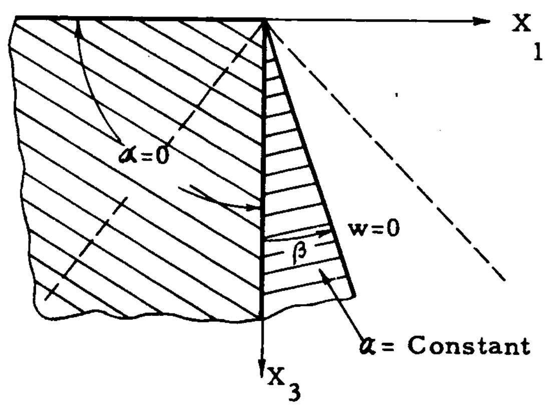
FIGURE 30. PLANFORM FOR EXAMPLE II

B. Example II

By the general principles stated in Section VI-A, one obtains the solution for Example II immediately from the solution for planform III of Section II and Example I of the present section.

C. Example III

The example illustrated in Figure 31 may not be solved by the above



Profile: Flat Plate, β is Counted Positive

FIGURE 31. PLANFORM FOR EXAMPLE III

methods: The case of the nonlifting wing with wedge planform is of course easily solved by superposition of solutions from Section III. But since the upper and lower surfaces are not independent, the solution for the flat profile may not be deduced from the solution for the wedge profile.

Fortunately, the boundary conditions for Example III are very similar to the conditions (3.29) for the correction function in Section III-H. It can easily be seen from the discussion in that section that the solution for Example III is given by Equation (3.30)

$$W = \frac{ik}{\pi}\left(\nu + \frac{1}{\nu}\right) \tag{6.1}$$

where k is a real number and

$$\nu = \sqrt{\eta}\ , \quad \eta = \frac{\epsilon - B}{1 - B\epsilon}, \ B = \frac{1 - \sqrt{1 - b^2}}{b}, \ b = \tan\beta$$

To make this formula complete, one has to establish the numerical relation between the constant k and the discontinuity in angle of attack at $\epsilon = 0$. This relation is obtained from Equation (1.42c).

$$\frac{dW}{d\epsilon} = \frac{ik}{\pi}\left(1 - \frac{1}{\nu^2}\right) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\eta}} \cdot \frac{(1 - B\epsilon) - (\epsilon - B)(-B)}{(1 - B\epsilon)^2} \tag{a}$$

$$\left(\frac{dW}{d\epsilon}\right)_{\epsilon=0} = \frac{ik}{\pi}\left[1 - \left(\frac{1}{-B}\right)\right] \frac{1}{2} \cdot \frac{1}{\sqrt{-B}} \cdot \frac{1 - B^2}{1}$$

$$= \frac{k}{\pi} \cdot \frac{1 - B}{\pm\sqrt{B}} \cdot \frac{1 + B^2 + 2B}{2B}$$

$$= \pm\frac{k}{\pi}\sqrt{\frac{2(1 - b)}{b}} \cdot \left(\frac{1}{b} + 1\right) \tag{b}$$

In evaluating Equation (b), Equations (1.37c) and (1.37i) were used, and the undetermined sign was introduced because of the square root. The correct sign of k may be determined in the final formula by the rule that w is positive for the positive value of α. Hence by Equation (b) and the discussion following Equation (1.42c)

Discontinuity in v = $\alpha\, w_\infty$

$$= \pm\pi i\left(-\frac{i}{2}\right)\frac{k}{\pi}\;\frac{(1-B)(1+B)^2}{2B\sqrt{B}} \qquad \text{(c)}$$

$$\pm\pi i\left(-\frac{i}{2}\right)\frac{k}{\pi}\sqrt{\frac{2(1-b)}{b}}\;\;\frac{b+1}{b}$$

Hence by the above rule for the determination of the sign of k

$$k = +\frac{b}{1+b}\sqrt{\frac{2b}{1-b}}\;\alpha w_\infty = \frac{4B\sqrt{B}\,\alpha\, w_\infty}{(1-B)(1+B)^2} \qquad (6.2)$$

if

$\nu = + i\sqrt{|\eta|}$ on top of wing

In proving Equation (6.2) one should remember that

$$\frac{1}{\nu} = \frac{1}{i\sqrt{|\eta|}} = \frac{-i}{\sqrt{|\eta|}}$$

is the dominant term in Equation (6.1).

Formulas (3.38) of Section III-I give the corresponding sidewash and downwash. Thus the complete solution for Example III is the following trio of functions

$$\frac{\pi U}{\alpha w_\infty} = \frac{-2\sqrt{B}\,(1+B^2)}{(1-B)(1+B)^2}\; i\left(\nu+\frac{1}{\nu}\right) - \ln\frac{(\nu - i\sqrt{B})\left(\nu+\frac{i}{\sqrt{B}}\right)}{(\nu + i\sqrt{B})\left(\nu-\frac{i}{\sqrt{B}}\right)} \qquad (6.3a)$$

$$\frac{\pi V}{\alpha w_\infty} = \frac{2\sqrt{B}}{1+B}\left(\frac{1}{\nu} - \nu\right) + \left[-i\ln\frac{(\nu - i\sqrt{B})\left(\nu-\frac{i}{\sqrt{B}}\right)}{(\nu + i\sqrt{B})\left(\nu+\frac{i}{\sqrt{B}}\right)} - \pi\right] \qquad (6.3b)$$

$$\frac{\pi W}{\alpha w_\infty} = \frac{4B\sqrt{B}}{(1-B)(1+B)^2}\; i\left(\nu+\frac{1}{\nu}\right) \qquad (6.3c)$$

Here

$$\nu = \sqrt{\frac{\epsilon - B}{1 - B\epsilon}}, \quad \frac{-2\sqrt{B}}{1 - B} \quad \frac{1 + B^2}{(1 + B)^2} = -\sqrt{\frac{2b}{1 - b}} \cdot \frac{1}{1 + b},$$

$$\frac{2\sqrt{B}}{1 + B} = \sqrt{\frac{2b}{1 + b}}, \quad \frac{4B\sqrt{B}}{(1-B)(1+B)^2} = \sqrt{\frac{2b}{1 - b}} \cdot \frac{b}{1 + b}$$

It is easily seen that v has the correct discontinuity at $\epsilon = 0$.

The evaluation of formulas (6.3) in physical coordinates in the plane of the wing can be made easily from Section III-I. In particular, the pressure on the wing is determined by

$$w = \frac{\alpha w_\infty}{\pi} \frac{2b\sqrt{b}}{1 + b} \sqrt{\frac{1 + t}{b - t}}, \quad -1 < t < b \qquad (6.4)$$

D. Example IV

This case is a generalization of the previous case as is seen from Figure 32. It is of course reduced to the previous case by the transformation

$$\epsilon \longrightarrow \frac{\epsilon - C}{1 - C\epsilon} = \theta, \quad C = \frac{1 - \sqrt{1 - c^2}}{c}, \quad c = \tan\gamma \qquad (d)$$

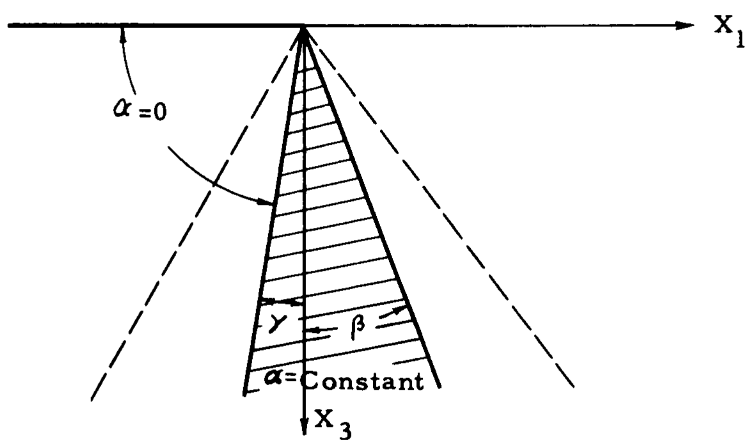


FIGURE 32. PLANFORM FOR EXAMPLE IV

The solution is obtained by combining formulas (1.57) and (6.3). One has to substitute θ for ϵ and $B^* = (B - C) / (1 - BC)$ for B in Equation (6.3) and then form linear combinations of these expressions according to Equation (1.57). The combination of these substitutions leaves the definition of ν in Equation (6.3) formally invariant, for ν^2 should now be $(\theta - B^*)/(1 - B^*\theta)$ which is simply $(\epsilon - B)/(1 - B\epsilon)$, as may be seen by direct computation. A more intuitive way of proving this invariance is given in the following paragraph:

ν^2 is the result of two successive homographic transformations of the special type discussed in Section I-L. The first one takes B into B*, and the second one B* into zero. Hence the composite transformation takes B into zero and has the form given above independent of C.

The procedure for obtaining the solution is straightforward and results will be given only for W:

$$\frac{\pi W}{\alpha w_\infty} = k_1 i \left(\nu + \frac{1}{\nu}\right) + \frac{2C}{1 - C^2} \ln \frac{(\nu - i\sqrt{B^*})\left(\nu + \frac{i}{\sqrt{B^*}}\right)}{(\nu + i\sqrt{B^*})\left(\nu - \frac{i}{\sqrt{B^*}}\right)} \tag{6.5}$$

$$\nu = \sqrt{\frac{\epsilon - B}{1 - B}}, \qquad B^* = \frac{B - C}{1 - BC}$$

$$k_1 = \frac{4B^*\sqrt{B^*}}{(1 - B^*)(1 + B^*)^2} \left[\frac{C(1 + B^{*2})}{B^*(1 - C^2)} + \frac{1 + C^2}{1 - C^2}\right].$$

$$= \frac{4B(1 - BC)}{(1 - C)(1 - B)(1 + B)^2} \sqrt{\frac{B - C}{1 - BC}}$$

or

$$k_1 = \frac{b^*}{1 + b^*} \sqrt{\frac{2b^*}{1 - b^*} \cdot \left(\frac{c}{b^*} + 1\right)} \frac{1}{\sqrt{1 - c^2}} = \frac{b}{1 + b} \sqrt{\frac{2(b - c)}{(1 - b)(1 - c)}}$$

From Equation (6.5) one obtains the following formula for the pressure on the upper surface of the wing:

$$w = \frac{\alpha w_\infty}{\pi}\left(\frac{2c}{\sqrt{1 - c^2}} \ln \left| \frac{\sqrt{(1 + c)(b - t)} - \sqrt{(b - c)(1 + t)}}{\sqrt{(1 - b)(c - t)}} \right| + \frac{2b}{1 + b}\sqrt{\frac{b - c}{1 - c}}\sqrt{\frac{1 + t}{b - t}}\right) \tag{6.6}$$

E. Note on Applications

One application is to find the lift of a twisted wing. The simplest example is a wing with rectangular planform and zero thickness where the slope is constant chordwise but varies spanwise. Such a wing may be obtained by superposition of wings of the type shown in Figure 29 with $\phi_1 = \phi_2 = \beta = 0$. However, it must be assumed that the slope of the wing is constant within the Mach cones from the leading edge tips. If this is not the case, the superposition described would give lift outside the wing. However, this lift may be removed by additional superposition of wings described in Section V. Another way of expressing this fact is to say that the twisted rectangular wing may be obtained by superposition of wings of the type shown in Figure 33.

If the solution for the wing in Figure 29 is applied to the wing in Figure 33, the region ABC will be lifting. This lift may be cancelled by superposition of wings described in Figure 22. These wings will then change the lift distribution in the region ABD of Figure 33.

The problem of the lift of a flat wing in a variable downwash field is of course similar to that of a twisted wing in a uniform flow field.

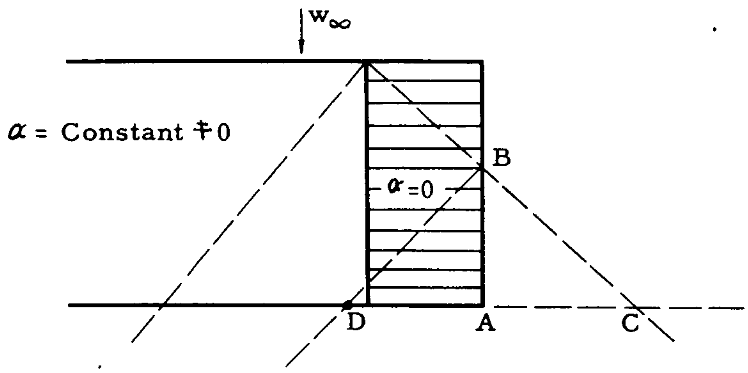


FIGURE 33. LIFTING RECTANGULAR WING ADJACENT TO FINITE FLAT PLATE

Finally consider the problem of finding the forces induced by the deflection of control surfaces. Two examples are shown in Figure 34.

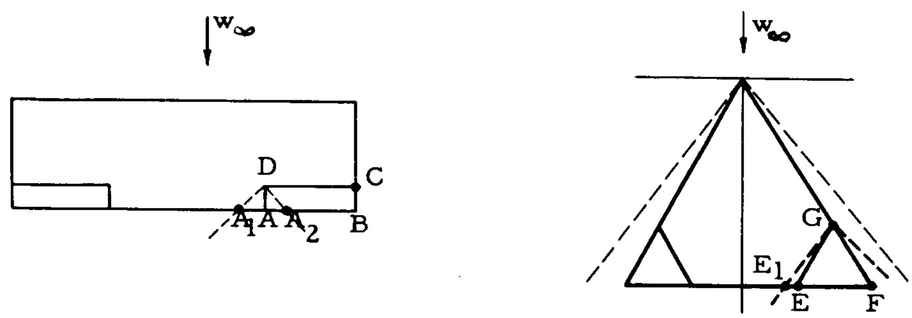


FIGURE 34. VARIOUS TYPES OF CONTROL SURFACES

It is clear that the effect of the control surface ABCD in the region DA_1A_2 is given by the solution for Example I in this section. However, normally this région contributes just a small part of the lift. On the main part of the control surface, two-dimensional theory applies. The effect of the control surface EGF in the region E_1FG is given by the solution for Example IV. In this case one is no longer dealing with a small correction. The main effect of deflecting the control surface is not given by any formula in the previous sections, but only by formula (6.6).

It has been assumed here that a gap which might exist along AD or EG in Figure 34 does not influence the pressure distribution appreciably.

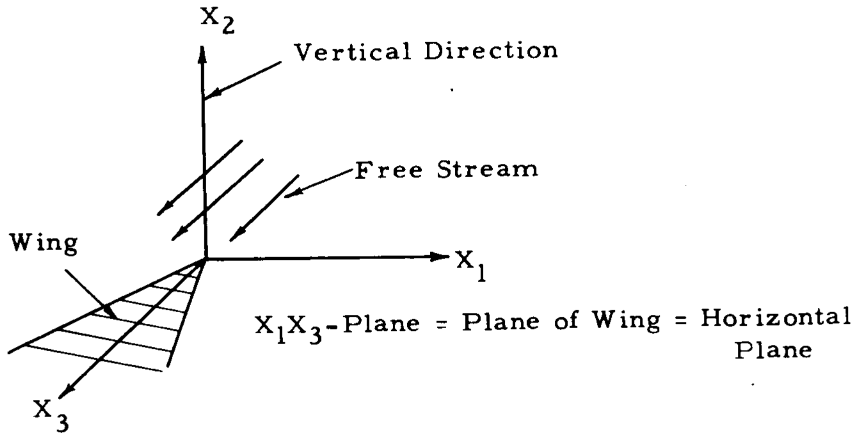
Further discussions of the applications may be found in Reference 3 where all solutions given in this section have been shown to have practical applications to control surfaces.

TABLE I

NOMENCLATURE

		Defined on page
p	pressure	4
p_∞	free-stream pressure	4
m	$\dfrac{1}{\sqrt{M^2 - 1}}$	4
M	free-stream Mach number	3
r	$\sqrt{x_1^2 + x_2^2}$	27
R	$Ts(r) = \dfrac{1 - \sqrt{1 - r^2}}{r}$	29
t	x_1/x_3	31
$Ts(a)$	Tschaplygin transformation $\dfrac{1 - \sqrt{1 - a^2}}{a}$	29
u	perturbation velocity in X_1-direction (sidewash)	4
u^*	harmonic conjugate of u	33
U	$u + iu^*$	35
v	perturbation velocity in X_2-direction (upwash)	4
v^*	harmonic conjugate of v	33
V	$v + iv^*$	33
w	perturbation velocity in X_3-direction	4
w^*	harmonic conjugate of w	33
w_∞	free-stream velocity	4
W	$w + iw^*$	33

TABLE I (cont'd)



		Defined on page
x	coordinate in ϵ-plane = $\mathrm{Re}(\epsilon)$	30
x_1, x_2, x_3	coordinates in physical space	
	x_1 in lateral direction	
	x_2 in vertical direction	
	x_3 in free-stream direction	
y	coordinate in ϵ-plane = $\mathrm{Im}(\epsilon)$	30
α	angle of attack	6
ϵ	$R \cdot e^{i\theta} = x + iy$	30
ζ	$r \cdot e^{i\theta} = x_1 + ix_2$	30
η	$\dfrac{\epsilon - B}{1 - B\epsilon}$	
θ	$\arctan \dfrac{x_2}{x_1}$	27

TABLE I (Cont'd)

		Defined on page
λ	half-angle of wedge	57
μ	Mach angle = arc tan m	54
ν	$\sqrt{\epsilon}$	79
ρ_∞	free-stream density	4
τ	arc tan t	31
ϕ	sweep-back angle	41
Φ	velocity potential	4
ψ	arc cos (tan ϕ)	42

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