NACA TN No. 1555

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1555

THE THEORETICAL LIFT OF FLAT SWEPT-BACK WINGS AT SUPERSONIC SPEEDS

By Doris Cohen

Ames Aeronautical Laboratory
Moffett Field, Calif.

Washington

March 1948

ERRATA

NACA TN 1555

THE THEORETICAL LIFT OF FLAT SWEPT-BACK
WINGS AT SUPERSONIC SPEEDS
By Doris Cohen
March 1948

The following list of changes, which includes the changes listed in all previously issued errata, should be noted:

Page 3, line 5. Read 0 instead of A.

Page 7, equation (6). The expression in the parentheses, last line, should read:

$$\left(\frac{m\sqrt{1-a_t^2}}{m-a_t} - 1\right)$$

Page 8, equation (8). The expression in brackets, last line, should read:

$$\left[m_t\sqrt{1-a_t^2} - (m_t-a_t)\right]$$

Page 13. The definition of ζ should read:

$$\zeta = \frac{2\beta\epsilon}{1+\beta^2\epsilon^2}$$

Page 14, line preceding equation (19). Correct expression for x_a to read:

$$x_a = \frac{\beta s}{a}$$

Page 17, line preceding equation (28). Read "wing" for "tip."

Page 18, equations (31) and (32). The right-hand sides should be multiplied by 2 (i.e., the number 2 appearing in the coefficients should be 4).

Page 20, equation (36). Insert β in the denominator of the coefficient of the right-hand member.

Page 20, bottom of page. The line preceding the series for $K(k')$ should read ". . . when k is small."

Page 23, equations (46) and (47). Multiply right-hand members by (-1).

Page 24, equation for $H'(m)$. Change to read:

$$\frac{m-a_t}{a_t^2 m^4(1+m)}\left[\frac{a_t}{m} - \frac{m-a_t}{2a_t(1+m)}\left\{1 - \left(1 + \frac{1}{2m}\right)\left[a_t + \frac{m-a_t}{6(1+m)}\right]\right\}\right]$$

Page 26, equation (51). Replace by

$$u = \text{r.p. } u_0 \frac{F(\varphi, \sqrt{1-m_t^2})}{K(\sqrt{1-m_t^2})}$$

where K and F are the complete and incomplete elliptic integrals of the first kind, of modulus $\sqrt{1-m_t^2}$, and

$$\varphi = \sin^{-1}\sqrt{\frac{1-t_0^2}{1-m_t^2}}$$

(This solution is due to H. S. Ribner, of the NACA.)

Page 27, equation (55). The expression in brackets should read:

$$\left[1 - \frac{\pi/2}{K(\sqrt{1-m_t^2})}\right]$$

Page 28, lines 2, 3, 4, and 5. The series expansion of $1 - \frac{2}{\pi}E$ is no longer pertinent. Substitute the following for lines 2, 3, 4, and 5:

The following series expansion (from reference 9) is useful:

$$1 - \frac{\pi/2}{K(\sqrt{1-m_t^2})} = \frac{\pi R}{2K(\sqrt{1-m_t^2})}\left[1 + \frac{R}{2(1+m_t)}\left(1 + \frac{1^2\ 3^2}{4^2}R^2 + \frac{1^2\ 3^2\ 5^2}{4^2\ 6^2}R^4 + \ldots\right)\right.$$

where $R = \dfrac{1-m_t}{1+m_t}$

Page 28, equation (57). Change to read:

$$\frac{\Delta L}{q\alpha} = \frac{-4m_t s^2}{a_t^2(m_t^2-m^2)E(\sqrt{1-m^2})}\left[\frac{m^2(m_t-a_t)^2}{\sqrt{m_t^2-m^2}}\left[\cos^{-1}\frac{m^2-m_ta_t}{m(m_t-a_t)} - \cos^{-1}\frac{m}{m_t}\right] -\right.$$

$$(m^2-m_ta_t)\sqrt{m^2-a_t^2}\left[1-\sqrt{\frac{m_t(1-a_t)}{m_t-a_t}}\right] -$$

$$\sqrt{m_t(1-m)(m_t+m)}\left\{\frac{1}{2}\left[(m+a_t)^2\frac{m_t-m}{m_t+m} + (m-a_t)^2\frac{1+m}{1-m}\right][F(\Phi_t,k) -\right.$$

$$\left.\left.F(\Phi_o,k)] - \frac{1}{m_t^2-m^2}[(m^2-m_ta_t)^2+3m^2(m_t-a_t)^2][E(\Phi_t,k)-E(\Phi_o,k)]\right\}\right]$$

Page 29, equation (58). Close brace at end of equation.

Page 30. The expression for the induced moment ΔM should read:

$$\Delta M = \frac{4\rho V\beta su_o}{K(\sqrt{1-m_t^2})}\int_{m_t}^{1}\left(\frac{m_t-a_t}{a_tm_t} + \frac{2}{3t_o}\right)F(\Phi,\sqrt{1-m_t^2})\frac{dS}{dt_o}\,dt_o$$

and

$$\frac{\Delta M}{q\alpha} = \frac{4m\beta s^3}{m_t^2E(\sqrt{1-m^2})}\left\{\frac{m_t-a_t}{a_t}\left[1-\frac{\pi/2}{K(\sqrt{1-m_t^2})}\right] + \frac{1}{3}\left[1-\frac{E(\sqrt{1-m_t^2})}{K(\sqrt{1-m_t^2})}\right]\right\} \quad (60)$$

Page 34, line 21. Amend to read "$(\Delta u)_a$ replaced by the right-hand member of equation (51)" and delete the expression which follows.

Page 38, following equation (74). Read "equation (38)" for "equation (30)."

Page 40, equation (77). The expression in brackets should read:

$$\left[\frac{\pi/2}{m_t K\left(\sqrt{1-m_t^2}\right)} - 1 \right]$$

Page 46. The table should read as follows:

Wing	Aspect ratio	C_{L_α}	C_D/C_L^2
Tapered wing	3.85	3.09	0.160
Untapered wing	1.73	1.97	.326
Modified untapered wing	2.45	2.62	.236

Page 49. In ϵ, read $\beta^2 z^2$ for βz^2

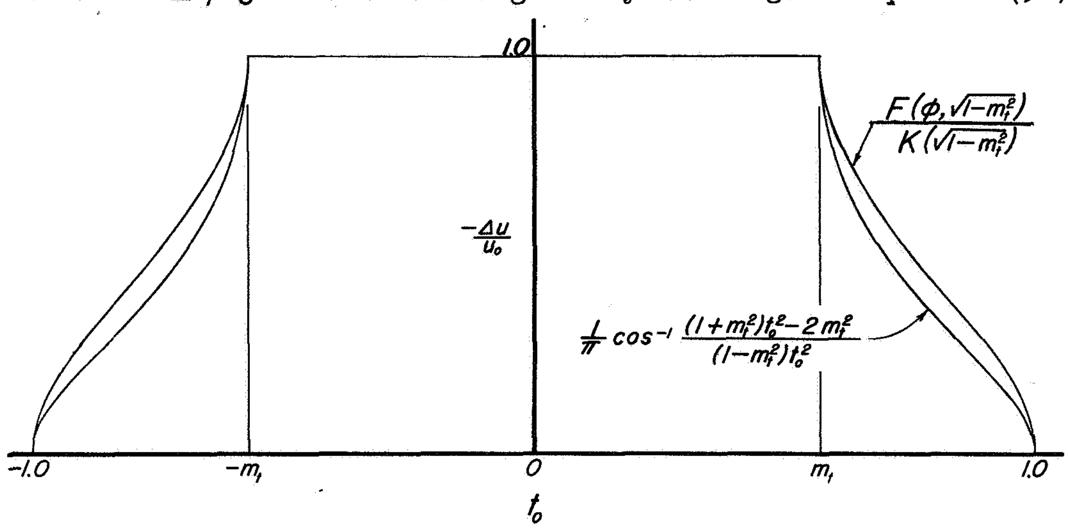
In $g(a)$, read $\frac{a}{m_t}$ for $\frac{a}{m}$

In Q, read $(1-m_t)$ for $(1+m_t)$

Page 52, equation (A6). Insert a_t^2 in denominator of the coefficient of the right-hand member.

Figures

The changes arising from the correction of equation (51) have only a small effect on the calculated results presented. (There is a major change in the downwash, which was not evaluated in this report.) The general shape of the pressure distribution remains the same. The magnitude of the correction is indicated in the figure below, in which the correct $\Delta u/u_o$ and the curve given by the original equation (51)



are compared for m_t=0.57. This figure corrects figure 13, and may be used to find the percent error in component 1 of the pressure distributions in figure 17. Component 5 in figure 17(d) will likewise be increased.

Tables I and II

Table I. The component of $\frac{L}{q\alpha}$ calculated for the untapered wing from equation (55) should be changed from −15.90 to −18.28, and from equation (73), from +0.14 to +0.16. The total $\frac{L}{q\alpha}$ is therefore 114.28 and C_{L_α}=1.97, or 0.034 per degree.

Table II. The values of $\frac{M}{q\alpha}$ contributed by the symmetrical wake solution, as corrected, are as follows: 19.9, 207.3, and 41.3. The amended totals are −457.4, −728.2, and −660.5, and the ratios of moment arms are 1.08, 0.82, and 1.01.

NACA-Langley - 8-8-49 - 1000

TABLE OF CONTENTS

Page

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1555

THE THEORETICAL LIFT OF FLAT SWEPT-BACK WINGS AT SUPERSONIC SPEEDS

By Doris Cohen

SUMMARY

The method of superposition of linearized conical flows has been applied to the calculation of the aerodynamic properties of thin, flat, swept-back wings at an angle of attack.

Various cases are distinguished, depending on the sweep of the leading and trailing edges relative to their respective Mach lines. Where the Mach line from the apex of the wing lies behind the leading edge but intersects the tip, formulas for the total lift are given in closed form. The induced drag is simply the lift times the angle of attack. To obtain the pitching moment the numerical integration of elementary functions is required. Where the leading edge, but not the trailing edge, is swept behind the Mach lines, the pressure distribution and the total lift are both given in closed form, but not the pitching moment. The induced drag is calculated by a simple formula from the lift.

The highly swept wing with both leading and trailing edges behind their respective Mach lines cannot be completely solved by the present method except by a series of successive approximations. However, the pressure distribution can be determined over all but the generally small region in which the tip and trailing-edge Mach cones overlap, and the total lift, pitching moment and induced drag can be approximated with practical accuracy, provided the aspect ratio is not so high nor the Mach number so low that the Mach lines from the trailing edge intersect the leading edge. (The latter case is not treated in the present paper.)

The complete procedure for the highly swept wing is outlined and various degrees of approximation suggested. Application of the general method to other plan forms is also discussed.

The investigation has produced an interesting result regarding the lift near the tip of a highly swept wing. The formulas show

an abrupt drop in lifting pressure across the tip Mach lines, the residual lift over most of the enclosed area being almost negligible. The attendant change in center-of-pressure location with Mach number may present a serious stability problem.

As a result of this development, a wing with tips normal to the stream is also investigated. This case is relatively simple to calculate and the necessary formulas are given in closed form. Calculations are presented for such a wing, as well as for two conventional swept-back wings with 63° sweep of the leading edge, at a Mach number of 1.5.

INTRODUCTION

Although a number of treatments of the problem of the lift on a thin flat surface in supersonic flow have recently been published (see bibliography of reference 1, and references 2 and 3) the lift distribution over a wing with both leading and trailing edges swept behind their respective Mach lines has not, at the present writing, been determined. The only explicitly formulated solutions (all based on the linearized form of the flow equations) are essentially of the so-called "conical" type, introduced by Busemann (reference 4); that is, solutions in which such quantities as the velocities and pressure are constant along lines radiating from a single point. The limitation is therefore automatically imposed that the boundary conditions to be satisfied must also be constant along such lines.[1] Under certain circumstances, conical fields may be superposed to give surfaces of nonconical plan form. Figure 1(a) shows one such case, a finite-span trapezoidal wing swept only slightly compared to the Mach lines. The basic solution here is that for a flat, symmetrical unyawed triangle at an angle of attack, and is conical with respect to the apex O. The triangle may extend to infinity. The pressure distribution over the shaded portion is constant; the values vary between the Mach lines from O, but are constant along any ray drawn through O. In order to obtain a finite wing, it is necessary to cancel the pressure beyond the tips AB and A'B' by superposing negatively loaded triangular surfaces with apexes at A and A' and one edge parallel to the stream. The conditions to be satisfied by the supplementary solutions are that (1) the pressure be constant over the subtracted surfaces in order to cancel the constant pressure in the basic solution, and (2) the downwash induced inboard of the streamwise edge be zero in order that the surface may remain flat. Both of these conditions are conical with respect to A or A', so that a conical solution may be used. The area behind

[1]The source-distribution methods of references 2 and 3, however, are applicable to curvilinear plan forms within certain restrictions.

BCB' may, of course, be subtracted without affecting the wing itself in any way, since the wing lies entirely ahead of the Mach lines originating from any point on the surface BCB'. Consider, however, the case shown in figure 1(b). Here the Mach number is such that the Mach lines from A intersect the tips of the wing. Starting with the same basic triangle, we find that the area to be removed outboard of A includes, in this case, a region over which the pressure varies, and is conical with respect to O. Since the boundaries of the region are, however, conical with respect to A, no one conical solution can satisfy the conditions. This case and the preceding one can be treated by the point-source method of reference 2.

It has become customary to describe a leading or trailing edge briefly as subsonic or supersonic accordingly as the component of free-stream velocity normal to it is subsonic or supersonic, that is, as its angle of sweep is greater or less than the sweep angle of the Mach lines. Thus wings 1(a) and 1(b) are said to have supersonic leading and trailing edges. In 1(c) is shown a wing with subsonic leading edge and supersonic trailing edge. In this case, as in 1(b), the pressure over the region outboard of the tip varies conically from the apex O; whereas the area involved is conical from A. The wing shown in 1(d), with which this paper is chiefly concerned, presents the same difficulties, not only at the tips, but also along its subsonic trailing edge.

A still higher degree of sweep relative to the Mach lines, or a higher aspect ratio, would result in the trailing-edge Mach lines intersecting the leading edge. This case, which introduces new problems, will not be covered in the present paper.

The treatment of the problem will be along the usual lines of the linearized theories. Since there have been so many careful examinations of the fundamental problem and equations of supersonic flow, these will be given only cursory mention where necessary to establish the course of the argument. The object of this paper will be to give a practical method for approximating the theoretical lift distribution for the conventional swept-back wing, and the emphasis will accordingly be on final formulas, where they can be given. The wing 1(d) with both leading and trailing edges swept behind their respective Mach lines, for reasons to be discussed later, can be treated only by successive approximations. The method of carrying out the successive steps will be indicated, but the amount of work required to find the pressure distribution beyond the first approximation appears prohibitive without the aid of a mechanical computer. The total lift and moment can be found to very good accuracy without much difficulty, however. Methods of approximating the effects beyond the first step will also be suggested.

The degree of approximation involved in the proposed estimates will depend on the aspect ratio and taper ratio of the wing. Numerical examples will be included in the paper and will serve to indicate the possible magnitude of the errors.

The method of canceling the pressure at the wing tips was originally suggested to the author by P. A. Lagerstrom of the California Institute of Technology. The treatment of the subsonic trailing edge is largely an extension of the same method of attack. Application of the methods to other than conventional swept wings is discussed in the present paper.

COORDINATES AND BASIC PARAMETERS

In accordance with the usual linearized approach, the boundary conditions are satisfied in the horizontal plane, rather than in the actual plane of the wing. The coordinate system will be chosen so that the apex of the leading edge is at the origin, the positive X-axis extending back in the stream direction along the axis of symmetry of the wing and the Y-axis perpendicular to the X-axis in the plane of the wing. The Z-axis is vertical, positive upward.

If the corresponding perturbation velocities are u, v, and w in the same order, the pressure difference, or local lift, is then, in the linearized theory,

$$\Delta p = 2\rho V u \tag{1}$$

or

$$\frac{\Delta p}{q} = 4\frac{u}{V}$$

where ρ is the density of the air, q the dynamic pressure, and V the velocity of the stream.

As previously noted, the conical solutions give values of the velocity u that are constant along radial lines. Therefore, the fundamental geometric quantities are the slopes of various radial lines. All such slopes are measured from the stream direction as the reference line. It is further convenient to relate these quantities to the inclination $1/\beta$ of the Mach lines in each problem, since it is by now well known that the solutions of the linearized supersonic flow equation are essentially functions of such a ratio.

Thus the following parameters and variables are defined:

$$m = \frac{\text{slope of leading edge}}{\text{slope of Mach lines}} = \beta \cot \Lambda$$

$$m_t = \beta \times \text{slope of trailing edge}$$

$a = \beta \times$ slope of any ray from the origin, or apex of the wing

$\beta = \sqrt{M^2-1}$, where M is the free-stream Mach number

Other symbols referring to relative angular quantities will be defined as needed in the same way. A summary of the symbols will be found in Appendix A.

I - BASIC LIFT DISTRIBUTION

WING WITH SUPERSONIC LEADING EDGE

Pressure Distribution Over Triangular Wing

The general procedure, as has been indicated, is to start with the solution for the infinite triangle having the same sweep of the leading edge as has the wing under investigation. This basic solution has only one parameter, the sweep of the leading edge relative to the Mach lines. Thus the incremental streamwise velocity distribution $u_\Delta(a)$ over a lifting triangular plate with supersonic leading edges is given (reference 5) by

$$\frac{u_\Delta(a)}{V\alpha} = \frac{2m}{\pi\beta\sqrt{m^2-1}} \cos^{-1} \sqrt{\frac{1-a^2}{m^2-a^2}} \tag{2}$$

where α is the angle of attack in radians. This function is plotted in figure 2.

Total Basic Lift

The integral of the pressure difference represented by this velocity distribution over the plan form of the wing under consideration will be called the basic lift L_o. The integration will be carried out for case 1(b), in which the leading-edge Mach lines cut across the tips. The element of area is a triangle formed by two rays of inclination a and $a + da$, and having as a third side either (1) the trailing edge of the wing or (2) the wing tip, accordingly, as a is less than or greater than a_t, the value corresponding to the ray through the tip of the trailing edge (fig. 2). Therefore

$$\frac{L_o}{q\alpha} = \frac{8}{\beta}\left[\int_0^{a_t} \frac{m_t^2 c_o^2}{2(m_t - a)^2} \frac{u_\Delta}{V\alpha}\, da + \int_{a_t}^{1} \frac{\beta^2 s^2}{2a^2} \frac{u_\Delta}{V\alpha}\, da + \int_1^{m} \frac{\beta^2 s^2}{2a^2} \frac{u_1}{V\alpha}\, da\right] \tag{3}$$

where c_o is the wing chord in the plane of symmetry, u_1 is the value of u_Δ for $a=1$ and s is the semispan of the wing. From the geometry of the wing we obtain

$$a_t = \frac{\beta s}{c_o + \beta s/m_t} \tag{4}$$

or

$$m_t c_o = \frac{\beta s}{a_t}\,(m_t - a_t)$$

The integration of equation (3) yields

$$\frac{L_o}{q\alpha} = \frac{4ms^2}{\pi\sqrt{m^2-1}} \left\{ \left[\frac{m_t}{a_t^2} + \frac{1}{m} - \frac{(m_t-a_t)^2}{a_t^2(m_t+m)}\right] \cos^{-1}\frac{1+ma_t}{m+a_t} \right.$$

$$+ \left[\frac{m_t}{a_t^2} - \frac{1}{m} - \frac{(m_t-a_t)^2}{a_t^2(m_t-m)}\right] \cos^{-1}\frac{1-ma_t}{m-a_t}$$

$$+ \frac{2m_t(m_t-a_t)^2\sqrt{m^2-1}}{a_t^2(m_t^2-m^2)\sqrt{m_t^2-1}} \left(\cos^{-1}\frac{1-m_ta_t}{m_t-a_t} - \cos^{-1}\frac{1}{m_t}\right)$$

$$\left. + \frac{2m^2(m_t-a_t)^2}{m_ta_t^2(m_t^2-m^2)} \cos^{-1}\frac{1}{m}\right\} \tag{5}$$

This form becomes indeterminate when $m_t=m$, that is, for an untapered wing. For this case

$$\frac{L_o}{q\alpha} = \frac{4s^2}{\pi a_t^2\sqrt{m^2-1}} \left\{\frac{(m+a_t)^2}{2} \cos^{-1}\frac{1+ma_t}{m+a_t} \right.$$

$$+ (m-a_t)\left[\frac{m+a_t}{2} - \frac{m(1-ma_t)}{m^2-1}\right] \cos^{-1}\frac{1-ma_t}{m-a_t}$$

$$- \frac{(m-a_t)^2(m^2-2)}{m^2-1} \cos^{-1}\frac{1}{m}$$

$$\left. + \frac{(m-a_t)^2}{\sqrt{m^2-1}}\left(\frac{m\sqrt{1-a_t^2}}{m-a_t} + 1\right)\right\} \tag{6}$$

Basic Moment

The center of pressure for this component of the lift is readily obtained when it is considered that each triangular element is uniformly loaded and therefore has its center of pressure two-thirds of the distance along the median back from its apex O. For $a < a_t$, the x coordinate of the center of pressure is

$$\frac{2m_t c_o}{3(m_t-a)} = \frac{2\beta s(m_t-a_t)}{3a_t(m_t-a)}$$

For $a > a_t$, the corresponding quantity is $\frac{2}{3}\frac{\beta s}{a}$. Then the basic pitching moment about the apex of the wing is given by

$$\frac{M_o}{q\alpha} = \frac{-8}{\beta}\left[\int_o^{a_t} \frac{{m_t}^3{c_o}^3}{3(m_t-a)^3}\frac{u_\Delta}{V\alpha}\,da + \int_{a_t}^{1} \frac{\beta^3 s^3}{3a^3}\frac{u_\Delta}{V\alpha}\,da + \int_1^m \frac{\beta^3 s^3}{3a^3}\frac{u_1}{V\alpha}\,da\right] \tag{7}$$

or

$$\frac{M_o}{q\alpha} = \frac{-8m\beta s^3}{3\pi {a_t}^3\sqrt{m^2-1}}\left\{\frac{1}{2}\left(m_t - \frac{{a_t}^3}{m^2}\right)\left(\cos^{-1}\frac{1+ma_t}{m+a_t} + \cos^{-1}\frac{1-ma_t}{m-a_t}\right)\right.$$

$$- \frac{(m_t-a_t)^3}{2}\left[\frac{1}{(m_t+m)^2}\cos^{-1}\frac{1+ma_t}{m+a_t} + \frac{1}{(m_t-m)^2}\cos^{-1}\frac{1-ma_t}{m-a_t}\right]$$

$$+ (m_t-a_t)^3\left[\frac{{m_t}^2+m^2}{({m_t}^2-m^2)^2} - \frac{1}{{m_t}^2}\right]\cos^{-1}\frac{1}{m}$$

$$+ \frac{(m_t-a_t)^3\sqrt{m^2-1}}{({m_t}^2-m^2)\sqrt{{m_t}^2-1}}\left(\frac{{m_t}^2+m^2}{{m_t}^2-m^2} + \frac{{m_t}^2}{{m_t}^2-1}\right)\left(\cos^{-1}\frac{1-m_t a_t}{m_t-a_t} - \cos^{-1}\frac{1}{m_t}\right)$$

$$\left. - \frac{(m_t-a_t)^2\sqrt{m^2-1}}{({m_t}^2-1)({m_t}^2-m^2)}\left[m_t\sqrt{1-a_t} - (m_t-a_t)\right] + \frac{{a_t}^3\sqrt{m^2-1}}{m^2}\cosh^{-1}\frac{1}{a_t}\right\} \tag{8}$$

For the untapered wing

$$\frac{M_o}{q\alpha} = \frac{-4m\beta s^3}{3\pi a_t{}^3\sqrt{m^2-1}} \left\{ \frac{3(m^2-a_t{}^2)(m+a_t)}{4m^2} \cos^{-1}\frac{1+ma_t}{m+a_t} \right.$$

$$+ (m-a_t)^3 \left[\frac{5}{4m^2} + \frac{3a_t}{m(m-a_t)^2} - \frac{m^2+2}{2(m^2-1)^2} \right] \cos^{-1}\frac{1-ma_t}{m-a_t}$$

$$+ (m-a_t)^3 \left[\frac{m^2+2}{2(m^2-1)^2} - \frac{2}{m^2} \right] \cos^{-1}\frac{1}{m}$$

$$+ \frac{2a_t{}^3\sqrt{m^2-1}}{m^2} \cosh^{-1}\frac{1}{a_t} - \frac{3(m-a_t)^3}{2(m^2-1)^{3/2}}$$

$$\left. + \frac{m-a_t}{2}\sqrt{\frac{1-a_t{}^2}{m^2-1}} \left[1 + \frac{(2m^2+1)(m-a_t)}{m(m^2-1)} \right] \right\} \tag{9}$$

WING WITH SUBSONIC LEADING EDGE

Pressure Distribution Over Triangular Wing

The solution for the flat lifting triangle with subsonic leading edges has been given by Stewart (reference 6) and others. In the notation of this paper the velocity distribution is given by

$$\frac{u_\Delta(a)}{V\alpha} = \frac{m^2}{\beta E\left(\sqrt{1-m^2}\right)} \times \frac{1}{\sqrt{m^2-a^2}} \tag{10}$$

where $E\left(\sqrt{1-m^2}\right)$ is the complete elliptic integral of the second kind, of modulus $\sqrt{1-m^2}$. It should be noted that for a given plan form the above expression and all those that follow in this section vary with Mach number only as the reciprocal of $E\left(\sqrt{1-m^2}\right)$.

Total Basic Lift

Integration of equation (10) over the wing plan form gives

$$\frac{L_o}{q\alpha} = \frac{4s^2}{a_t^2E\left(\sqrt{1-m^2}\right)}\left[\frac{m^2(m_t-a_t)^2}{m_t^2-m^2}\left\{\frac{m_t}{\sqrt{m_t^2-m^2}}\left[\cos^{-1}\frac{m^2-m_ta_t}{m(m_t-a_t)} - \cos^{-1}\frac{m}{m_t}\right]\right.\right.$$

$$\left.\left. + \frac{m}{m_t}\right\} - \frac{m_t(m^2-m_ta_t)}{m_t^2-m^2}\sqrt{m^2-a_t^2}\right] \qquad (11)$$

As before, a special form is required when $m_t = m$:

$$\frac{L_o}{q\alpha} = \frac{4s^2}{a_t^2E\left(\sqrt{1-m^2}\right)}\left\{\frac{(m-a_t)^2}{3}\left[\frac{2m-a_t}{(m-a_t)^2}\sqrt{m^2-a_t^2} - 2\right] + a_t\sqrt{m^2-a_t^2}\right\} \qquad (12)$$

Basic Moment

The moment is obtained from equation (7) with, however, the third integral deleted, since $m < 1$, and with m substituted for 1 as the upper limit of the second integral. Setting $\frac{u\Delta}{V\alpha}$ equal to the expression given by equation (10) we obtain

$$\frac{M_o}{q\alpha} = \frac{-4m^2\beta s^3}{3a_t^3E\left(\sqrt{1-m^2}\right)}\left[\frac{(m_t-a_t)^3}{(m_t^2-m^2)}\left\{\frac{2m_t^2+m^2}{(m_t^2-m^2)^{3/2}}\left[\cos^{-1}\frac{m^2-m_ta_t}{m(m_t-a_t)}\right.\right.\right.$$

$$\left.\left. - \cos^{-1}\frac{m}{m_t}\right] + \frac{3m}{m_t^2-m^2} + \frac{m}{m_t^2}\right\}$$

$$\left. - \frac{m_t}{m_t^2-m^2}\left[\frac{m^2-m_ta_t}{m^2} + \frac{3(m_t-a_t)^2}{m_t^2-m^2}\right]\sqrt{m^2-a_t^2} + \frac{a_t^3}{m^3}\cosh^{-1}\frac{m}{a_t}\right] \qquad (13)$$

For the untapered wing the corresponding formula is

$$\frac{M_o}{q\alpha} = -\frac{8\beta s^3}{3a_t{}^3E\left(\sqrt{1-m^2}\right)}\left\{\frac{1}{3}\left[(m+a_t) + \frac{a_t}{10} + \frac{2(m-a_t)^2}{5m}\right]\sqrt{m^2-a_t{}^2}\right.$$

$$\left. - \frac{7(m-a_t)^3}{15m} + \frac{a_t{}^3}{2m}\cosh^{-1}\frac{m}{a_t}\right\} \tag{14}$$

II – TREATMENT OF THE TIP

In canceling the pressures at and beyond the tip, the problem is brought within the limitations of the conical solutions by breaking the region down into an infinite number of constantly loaded <u>overlapping</u> sectors of infinite extent. (See fig. 2.) These sectors are bounded on one side by the wing tip; the second side is the extension of a ray from the apex 0 of the wing. The (constant) pressure on each sector is then $2\rho V\frac{du_\Delta}{da}da$, where a is the previously defined parameter β times the inclination of the ray from 0, the ray in this case being the particular one bounding the element.

Any element bounded by a ray of inclination a/β will hereinafter be referred to as the element a, or sector a. The apex of the sector will be designated by the coordinates x_a, y_a and the value of the constant incremental velocity over the element by u_a.

ELEMENTARY SOLUTION FOR TIP

The flow fields to be superposed on the basic lifting triangle to cancel lift beyond the tips will be required to satisfy the following conditions:

1. The streamwise velocity u must be constant over the sector between the tip and the specified ray.

2. The velocity u must be equal in value but opposite in sign on the upper and lower surfaces of the sector, in order to produce lift.

3. The streamwise velocity u must be zero outside the sector on the free stream side.

4. The associated downwash or vertical velocity w must be zero outside the sector on that side adjoining the wing, in order not to disturb the boundary condition of constant slope of the wing already satisfied by the basic solution.

5. All induced velocities must go to zero on the Mach cone from the apex of the sector. For a discussion of this condition, see reference 7.

6. The velocity components u, v, and w must satisfy the linearized supersonic flow equation and the equations for irrotationality.

The solution of the supersonic flow equations satisfying the specified boundary conditions (1 through 5) has been derived by Lagerstrom, using the method of analytic extension. No attempt will be made to reproduce the derivation here. The discussion which follows is included only to illustrate the nature of the solution.

Condition (6) may be shown to be satisfied by any analytic function of the complex variable

$$\epsilon = \frac{y+iz}{x+\sqrt{x^2-\beta^2y^2-\beta^2z^2}} \tag{15}$$

It is convenient to set up the solution for u as such a function of ϵ, since the boundary conditions are largely specified in terms of u. Condition 4, a condition on w, can also be related to u by the following procedure:

Let u represent the real part of the complex analytic function $U(\epsilon) = u(x,y,z) + i\bar{u}(x,y,z)$ and let the corresponding complex potential be

$$\Phi = \int^x U\,dx = \varphi+i\bar{\varphi}$$

Then $w = \partial\varphi/\partial z$ will be the real part of $W = \partial\Phi/\partial z$. We may write

$$W = \frac{\partial}{\partial z}\int^x U\,dx$$

as

$$W = \int^x \frac{\partial U}{\partial z}\,dx \tag{16}$$

the additional terms arising in the differentiation under the integral sign being zero.

The lower limit of the integral is arbitrary as far as the determination of Φ is concerned, since it can only change Φ by a constant and its derivatives not at all. However, in order that condition 5 be satisfied it now appears necessary to specify a point on the Mach cone as the limit.

Equation (16) may now be written

$$W = \int_{M.C.}^{x} \left(\frac{dU}{d\epsilon}\frac{\partial\epsilon}{\partial z}\right)\left(\frac{d\epsilon}{\partial\epsilon/\partial x}\right) = \int_{M.C.}^{x} \frac{\partial\epsilon/\partial z}{\partial\epsilon/\partial x}\, dU \tag{17}$$

The ratio of partial derivatives appearing under the integral sign is independent of the function assumed for U. In the plane $z=0$, it is equal to

$$\frac{-i\sqrt{x^2-\beta^2y^2}}{y} \quad \text{(See footnote 2.)}$$

Inside the Mach cone $x > \beta y$ and $\frac{\partial\epsilon/\partial z}{\partial\epsilon/\partial x}$ is an imaginary quantity. It follows that over any region of the X–Y plane in which U has no imaginary part, the real part of W will remain constant. Since the region to which condition 4 applies is bounded by the Mach cone, on which W is zero, it follows that condition 4 will be satisfied if the imaginary part of U is zero over the region.

It is convenient now to introduce the variable

$$\zeta = \frac{2\beta}{1+\beta^2\epsilon^2}$$

which will be seen to reduce to $a = \beta\frac{y}{x}$ when $z=0$.

---

[2]The general expression for $\frac{\partial\epsilon/\partial z}{\partial\epsilon/\partial x}$ includes a real part which becomes indeterminate, rather than zero, when both y and z equal zero, and the argument that follows no longer holds. However, the X–axis, along which this occurs, is not included in the region under discussion (see condition 4), so that the reasoning is adequate.

Without rewriting equation (15), we will consider the variable ϵ to be referred to the apex (x_a, y_a) of the elementary sector a as origin and define the new angular parameter (corresponding to a in the fixed reference coordinates)

$$t_a = \beta \frac{y-y_a}{x-x_a} \tag{18}$$

This will then be the value of ζ in the plane of the wing. The ray t_a associated with any point $P(x,y)$ and a particular sector a is shown in figure 2. When the solution is being applied to the region of the wing tip, $y_a = s$ and $x_a = \frac{s}{a}$ so that

$$t_a = \beta \frac{y-s}{x-\beta s/a} \tag{19}$$

In the region of the right-hand tip, t_a is negative on the wing, and positive on the elementary sector. It takes on the following special values:

$t_a = a$ on the free-stream boundary of the sector

$t_a = o$ along the wing tip

$t_a = \pm 1$ along either Mach line

Conditions 1 through 4 may now be written

$$\left.\begin{matrix}1.\\2.\end{matrix}\right\} \quad o \leq t_a \leq a, \lim_{z \to 0} u(x,y,z) = \frac{z}{|z|} u_a \; (u_a \text{ constant})$$

3. $a < t_a \leq 1,\; u(x,y,o) = 0$

4. $-1 < t_a < o,\; \bar{u}(x,y,o) = 0$

Consider now the complex function

$$U = \frac{u_a}{\pi} \cos^{-1} \frac{a+\zeta+2a\zeta}{\zeta-a} = \frac{u_a}{\pi}\, i \log \frac{a+\zeta+2a\zeta+2\sqrt{a\zeta(1+a)(1+\zeta)}}{\zeta-a} \tag{20}$$

As z approaches zero through positive and negative values, ζ can be seen to approach the real value t_a through conjugate complex values. The argument of the logarithm in equation (20) will also take on conjugate complex values, as will the logarithm itself. The real part of U, however, is proportional to the imaginary part of the logarithm and will therefore differ in sign above and below the X-Y plane, as required by condition 2. In the plane $z=0$,

$$u = \pm \text{ r.p. } \frac{u_a}{\pi} \cos^{-1} \frac{a+t_a+2at_a}{t_a-a} \tag{21}$$

The argument of the inverse cosine in equation (21) is plotted against t_a in figure 3. When the argument is greater than 1 or less than − 1 the function U is complex. Both the real part u and the imaginary part $\bar{u}$ are plotted, but only for the upper sign in equation (21).

It will be observed that conditions 1, 2, and 3 are satisfied by u and condition 4 by the corresponding imaginary function $\bar{u}$. By equation (16) and the corresponding expression for the lateral velocity component, w and v are determined, regardless of the function assumed for u, in such a way as to satisfy condition 5. That u also satisfies that condition is readily demonstrated, since on the Mach cone $x = \beta\sqrt{y^2+z^2}$ and $\zeta = x/\beta y > 1$. When ζ is greater than 1, the argument $\frac{a+\zeta+2a\zeta}{\zeta-a}$ is positive and greater than one, so that U (equation (20)) is imaginary, or $u=0$.

APPLICATION TO WING WITH SUPERSONIC LEADING EDGE

As long as the wing tapers in the usual sense, that is, from root to tip, the condition of a supersonic leading edge implies a supersonic trailing edge as well. In that case the pressure in the wake may be canceled without inducing any pressures on the wing ahead. The section that follows will therefore cover completely all corrections to the basic lift for a swept-back wing of type 1(b). Moreover when the total lift has been found, the drag due to lift is also known in this case, since it is merely the lift times the angle of attack when the leading edge of the wing is supersonic.

Induced Pressure Distribution

The basic lift distribution has been given by equation (2). Following the procedure outlined in the preceding section, we divide the lift outboard of the wing tip along projections of rays from the apex of the wing (fig. 2). Beyond $a=1$ no further division is necessary, since the basic pressure is constant at u_1 ahead of the Mach line from O. Thus the solution to be superposed may be considered to be made up of the sector $a=m$ with constant incremental velocity $u_a = -u_1$ and an infinite number of narrower sectors $(a<1)$ each with constant infinitesimal velocity

$$u_a = \frac{du_\Delta}{da}\, da \tag{22}$$

Then the total induced correction to the basic velocity distribution is

$$\Delta u = \frac{-u_1}{\pi} \cos^{-1} \frac{m+t_m+2mt_m}{t_m-m} + \frac{1}{\pi}\int_{a_o}^{1} \frac{du_\Delta}{da} \cos^{-1} \frac{a+t_a+2at_a}{t_a-a}\, da \tag{23}$$

where t_a is the function of x,y and a defined by equation (19), and must be so expressed before integrating; and where t_m is the function of x and y obtained by setting a equal to m in t_a. The lower limit a_o of the integral corresponds to the most rearward sector that can influence the pressure of the point x,y; that is, the sector such that the Mach line from its apex x_a,s passes through the point x,y. The expression for a_o, found by setting t_a equal to -1 and solving for a, is

$$a_o = \frac{\beta s}{x+\beta(y-s)} \tag{24}$$

This parameter will be additionally useful as the value of a at which the function given by equation (21) goes to zero and its derivative has a singularity. If a_o is equal to or greater than 1, the induced velocity Δu reduces to the first term of equation (23).

From equation (2)

$$u_1 = \frac{mV\alpha}{\beta\sqrt{m^2-1}} \tag{25}$$

and

$$\frac{du_\Delta}{da} = \frac{2mV\alpha}{\pi\beta} \frac{a}{(m^2-a^2)\sqrt{1-a^2}} \tag{26}$$

The integration of equation (23) for the local pressure has not been carried out, since the solution for this case is obtained more easily by the method of reference 2. However, the total lift induced on the wing by canceling pressure outboard of the tip is readily obtained by the present method if the order of integration is reversed and the total lift induced by a single element found first.

Total Induced Lift

The change in lift due to any tip element a, obtained from the velocity distribution (equation (21)), is

$$(\Delta L)_a = 2\rho V \int_{-1}^{0} \frac{u_a}{\pi} \cos^{-1} \frac{a+t_a+2at_a}{t_a-a} \frac{dS}{dt_a} dt_a \tag{27}$$

where dS is the element of wing area. If the line $t_a = -1$ does not cut the opposite tip,

$$\frac{dS}{dt_a} = \frac{\beta m_t{}^2 s^2}{2} \left(\frac{1}{a_t} - \frac{1}{a}\right)^2 \frac{1}{(m_t-t_a)^2} \tag{28}$$

where a_t is the value of a defined by equation (4). Then equation (27) becomes, after integration by parts,

$$(\Delta L)_a = \frac{\rho V m_t{}^2 \beta s^2}{a_t{}^2 a^2} u_a \quad (a_t-a)^2 \; g(a) \tag{29}$$

where

$$g(a) = \frac{1}{m_t-a} \left(\frac{\sqrt{a+a^2}}{\sqrt{m_t+m_t{}^2}} - \frac{a}{m_t}\right)$$

and the total change in lift is (taking both wing tips into account)

$$\Delta L = 2\left[-\ (\Delta L)_{a=m} + \frac{\rho V m_t{}^2 \beta s^2}{a_t{}^2} \int_{a_t}^{1} g(a) \frac{du_\Delta}{da} \frac{(a_t-a)^2}{a^2} da \right] \tag{30}$$

For the first term of equation (30), a is taken equal to m in equation (29), and u_a equal to u_1 (equation (25)). The second term is integrable by elementary methods. The final result is

$$\frac{\Delta L}{q\alpha} = \frac{2m_t^2 s^2 m}{\pi a_t^2\sqrt{m^2-1}} \left[\frac{(m-a_t)^2}{m(m_t-m)}\left(\frac{1}{m_t}\cos^{-1}\frac{1-ma_t}{m-a_t} - \frac{1}{m}\sqrt{\frac{m+m^2}{m_t+m_t^2}}\cos^{-1}\frac{m-2ma_t+a_t}{m-a_t}\right)\right.$$

$$-\frac{(m+a_t)^2}{m(m_t+m)}\left(\frac{1}{m_t}\cos^{-1}\frac{1+ma_t}{m+a_t} - \frac{1}{m}\sqrt{\frac{m^2-m}{m_t+m_t^2}}\cos^{-1}\frac{-m+2ma_t+a_t}{m+a_t}\right)$$

$$\left.-\frac{2(m_t-a_t)^2\sqrt{m^2-1}}{m_t(m_t^2-m^2)\sqrt{m_t^2-1}}\left(\cos^{-1}\frac{1-m_t a_t}{m_t-a_t} - \cos^{-1}\frac{m_t-2m_t a_t+a_t}{m_t-a_t}\right)\right] \qquad (31)$$

As before, a special form is needed for the untapered wing:

$$\frac{\Delta L}{q\alpha} = \frac{2ms^2}{\pi a_t^2\sqrt{m^2-1}} \left\{(m-a_t)\left[\frac{(3m^2-1)(m-a_t)}{2m(m^2-1)} - 2\right]\cos^{-1}\frac{1-ma_t}{m-a_t}\right.$$

$$-(m-a_t)\left[\frac{(3m-2)(m-a_t)}{2m(m-1)} - 2\right]\cos^{-1}\frac{m-2ma_t+a_t}{m-a_t}$$

$$-\frac{(m+a_t)^2}{2m}\left(\cos^{-1}\frac{1+ma_t}{m+a_t} - \sqrt{\frac{m-1}{m+1}}\cos^{-1}\frac{-m+2ma_t+a_t}{m+a_t}\right)$$

$$\left.+\frac{(m-a_t)\sqrt{1-a_t}}{\sqrt{m-1}}\left(\sqrt{\frac{a_t}{m}} - \frac{\sqrt{1+a_t}}{\sqrt{m+1}}\right)\right\} \qquad (32)$$

Induced Moment

The moment due to this correction to the lift may also be expressed in closed form, but the formulas are so long and unwieldy that their integration in general terms was not considered worthwhile. The moment arm of any element dt_a from x_a,s about the Y-axis is

$$\beta s\left[\frac{1}{a} + \frac{2m_t}{3(t_a-m_t)}\left(\frac{1}{a} - \frac{1}{a_t}\right)\right]$$

The moment induced by any element a is

$$(\Delta M)_a = \frac{-\rho V\beta^2 s^3 m_t^2}{a_t^2 a^3} (a-a_t)^2 \; u_a \left[g(a) + \frac{a-a_t}{3a_t} h(a) \right] \tag{33}$$

where

$$h(a) = \left(1 + \frac{m_t}{m_t-a}\right) g(a) - \frac{1}{2(1+m_t)(m_t-a)} \sqrt{\frac{a+a^2}{m_t+m_t^2}}$$

The total induced moment (including both wing tips) is

$$\Delta M = -2 \left\{ (\Delta M)_{a=m} + \frac{\rho V\beta^2 s^3 m_t^2}{a_t^2} \int_{a_t}^{1} \frac{(a-a_t)^2}{a^3} \left[g(a) + \frac{a-a_t}{3a_t} h(a) \right] \frac{du_\Delta}{da} da \right\} \tag{34}$$

As in the case of the lift, the first term is evaluated by setting a equal to m and u_a equal to u_1 in equation (33). In the case that $m_t=m$, special limiting forms of g and h will be required, namely,

$$g(m,m) = \frac{1}{2m(1+m)}$$

and

$$h(m,m) = \frac{1}{m(1+m)} \left[1 - \frac{3}{8(1+m)} \right]$$

Integration of the second term can be done numerically for any particular case.

APPLICATION TO WING WITH SUBSONIC LEADING EDGE

The case of a wing with subsonic leading edge $(m<1)$ requires slightly different treatment because the pressure at the leading edge is infinite. The formulas to be obtained for this case will, however, give the complete theoretical solution for such a wing as long as the trailing edge is not also subsonic. Where this condition does not hold $(m_t<1)$, the following sections will give the first step in the complete solution.

Induced Pressure Distribution

It is necessary to write equation (23) for the induced velocity at x,y as

$$\Delta u = \lim_{\lambda \to m} \left[\frac{-u_\Delta(\lambda)}{\pi} \cos^{-1} \frac{\lambda + t_\lambda + 2\lambda t_\lambda}{t_\lambda - \lambda} + \frac{1}{\pi} \int_{a_0}^{\lambda} \frac{du_\Delta}{da} \cos^{-1} \frac{a + t_a + 2at_a}{t_a - a} \, da \right] \tag{35}$$

where, as before, t_a must be replaced by a function of x, y, and a, (equation (19)). Integration of the second term by parts gives rise to a term which exactly cancels the first term and is zero at the other limit, leaving after substitution for u_Δ,

$$\frac{\Delta u}{V\alpha} = \frac{m^2 (x+\beta y)}{\pi E(\sqrt{1-m^2})} \sqrt{\frac{a_0(s-y)}{s}} \int_{a_0}^{m} \frac{da}{(\beta y - ax)\sqrt{(m^2-a^2)(1+a)(a-a_0)}} \tag{36}$$

This integral is finite and can be evaluated in terms of elliptic integrals as follows:

$$\frac{\Delta u}{V\alpha} = \frac{m^{3/2}}{\pi E(\sqrt{1-m^2})\beta} \left[\sqrt{\frac{2\beta(s-y)}{x+\beta y}} K(k) \right.$$
$$\left. -2x \sqrt{\frac{m}{m^2x^2 - \beta^2y^2}} \left\{ \frac{F(\psi,k')}{K(k')} \left[\frac{\pi}{2} - K(k)E(k') \right] + K(k)E(\psi,k') \right\} \right] \tag{37}$$

where the modulus $k = \sqrt{\dfrac{(m-a_0)(1-m)}{2m(a_0+1)}}$, $k' = \sqrt{1-k^2}$

and the argument $\psi = \sin^{-1} \sqrt{\dfrac{a_0(mx+\beta y)}{\beta s(a_0+m)}}$. The elliptic integrals F, K, and E are extensively tabulated in reference 8. They are also available in Pierce's Table of Integrals and most handbooks, but experience has shown that k' is generally very nearly unity and that $K(k')$ as tabulated in the handbooks is not always satisfactory for interpolation. The following series (reference 9) converges very rapidly and is preferable for computing $K(k')$ when K is small:

$$K(k') = -\log\frac{k}{4} - \frac{1}{4}\left(\log\frac{k}{4} + 1\right)k^2 - \frac{9}{64}\left(\log\frac{k}{4} + \frac{7}{6}\right)k^4 - \ldots$$

At the tip, $y=s$ and the first term in equation (37) vanishes. In the second term, ψ becomes $\frac{\pi}{2}$ and $E(\psi,k')$ and $F(\psi,k')$ reduce to the complete integrals $E(k')$ and $K(k')$, respectively.

The induced velocity correction Δu will then be seen to be exactly equal to $-u_\Delta$, bringing the lift to zero at the wing tip.

Drop in Lift Across Tip Mach Line

An interesting effect shows itself at the other limit of the tip region – the Mach line from the tip of the leading edge. Along this line only the influence of the leading-edge pressure is felt; that is, $a_o=m$. Then $k=0$, $k'=1$, $K(k) = \frac{\pi}{2}$. $K(k') = \infty$ and $E(k')=1$.

$E(\psi,k')$ reduces to $\sin\psi = \sqrt{\frac{mx+\beta y}{2\beta s}}$ and finally

$$\frac{\Delta u}{V\alpha} = \frac{-m}{E\left(\sqrt{1-m^2}\right)}\,\frac{s}{\sqrt{2(x+\beta y)(mx-\beta s)}} \tag{38}$$

This result corresponds to a finite drop in pressure across the Mach line from the tip, an effect which does not appear as long as the leading edge is ahead of the Mach lines. The magnitude of the drop relative to the pressure just ahead of the Mach line should be of interest. This ratio is

$$\frac{\Delta u}{u_\Delta} = \frac{-\beta s}{m}\sqrt{\frac{(m^2-a^2)}{2(x+\beta y)(mx-\beta s)}}$$

Since $a=\beta y/x$, and on the Mach line from the tip $\beta(s-y) = x - \frac{\beta s}{m}$, the ratio $\Delta u/u_\Delta$ can be rewritten very simply in terms of m and a as follows:

$$\frac{\Delta u}{u_\Delta} = -\sqrt{\frac{(1+a)(m+a)}{2m(1+m)}} \tag{39}$$

This function is plotted in figure 4 against a/m.

It will be seen that the percentage of loss of lift at the tip is very large and, in fact, that for any but the lowest aspect ratio wings the lift remaining in that region, which drops to zero at the tip itself, is almost negligible. Some indication of this effect is contained in the results of reference 10 for the limiting case of $m=0$. It is an effect of considerable practical interest, suggesting as it does a considerable change in pitching moment with Mach number as well as the inefficiency of the tip area of the wing.

Total Induced Lift

Proceeding to the calculation of the total correction to the lift, we integrate the change in lift $(\Delta L)_a$ (equation (29)) induced by each element a, over the range $a_t \leq a \leq m$. Again we encounter the difficulty that $u_\Delta \rightarrow \infty$ as $a \rightarrow m$ and must therefore write the total lift in terms of limiting values. For convenience the function

$$G(a) = \left(\frac{1}{a} - \frac{1}{a_t}\right)^2 g(a) \tag{40}$$

from equation (29), is defined. Then the total induced lift, corresponding to equation (30) for the preceding case, may be written

$$\Delta L = 2\rho V m_t^2 \beta s^2 \lim_{a \to m} \left[-u_\Delta(a) G(a) + \int_{a_t}^{a} \frac{du_\Delta}{da} G(a) da \right] \tag{41}$$

Integrating by parts results in cancelation of the first term inside the brackets. Since $G(a_t)$ is zero, equation (41) reduces to

$$\Delta L = -2\rho V m_t^2 \beta s^2 \int_{a_t}^{m} u_\Delta(a) G'(a)\, da \tag{42}$$

Equation (42) has been integrated in terms of elliptic integrals, but because the result involves several new functions it was thought better to present it in an appendix (Appendix B). For practical use graphical or numerical integration may be preferred. Here again, however, the difficulty arises at the leading edge that u_Δ and therefore the integrand of (42) becomes infinite, although the integral is finite. For numerical integration it is possible to dispose of the singularity by integrating once more by parts. A somewhat simpler method is to write

$$\Delta L = -2\rho V m_t^2 \beta s^2 \left\{ \int_{a_t}^{m} u_\Delta(a) \left[G'(a) - G'(m) \right] da + \int_{a_t}^{m} u_\Delta(a)\, G'(m) da \right\}$$

The integrand of the first term can now be shown to approach zero as a approaches m, and the second term is readily integrable, since $G'(m)$ is a constant. The result, when u_Δ has been replaced by its value from equation (10) and the second term integrated, is

$$\frac{\Delta L}{q\alpha} = \frac{-4m^2m_t^2s^2}{E\left(\sqrt{1-m^2}\right)}\left[G'(m)\cos^{-1}\frac{a_t}{m} + \int_{a_t}^{m}\frac{G'(a)-G'(m)}{\sqrt{m^2-a^2}}\,da\right] \tag{43}$$

$$G'(a) = \frac{a-a_t}{a_t^2a^2}\left[\left(\frac{a_t}{a} + \frac{m_t-a_t}{m_t-a}\right)g(a) - \frac{a-a_t}{2(m_t-a)\sqrt{(m_t+m_t^2)(a+a^2)}}\right] \tag{44}$$

When $m_t=m$, $G'(m)$ reduces to

$$\frac{m-a_t}{a_tm^4(1+m)}\left[1 + \frac{m-a_t}{8a_t(1+m)}\right]$$

Induced Moment

The moment induced by cancellation of the tip pressure is found, as in the preceding case, by the integration of $(\Delta M)_a$ (equation (33)) over the appropriate range of a. Again, the actual integration is best done numerically for any specific case in which the moment may be required. Because u_a becomes infinite at the leading edge, it will first be necessary to integrate by parts, as in the case of the total lift. Following the same procedure, we let

$$H(a) = \frac{1}{a}\left(\frac{1}{a_t} - \frac{1}{a}\right)^2\left[g(a) + \frac{a-a_t}{3a_t}h(a)\right] \tag{45}$$

from equation (33), so that the total induced moment is

$$\Delta M = 2\rho V\beta^2s^3m_t^2\lim_{a\to m}\left[-u_\Delta(a)\,H(a) + \int_{a_t}^{a}\frac{du_\Delta}{da}H(a)da\right]$$

$$= -\,2\rho V\beta^2s^3m_t^2\int_{a_t}^{m}u_\Delta(a)H'(a)da \tag{46}$$

For numerical integration, the procedure outlined for finding the lift is repeated, giving

$$\frac{\Delta M}{q\alpha} = \frac{-4m^2m_t^2s^3\beta}{E\left(\sqrt{1-m^2}\right)}\left[H'(m)\cos^{-1}\frac{a_t}{m} + \int_{a_t}^{m}\frac{H'(a)-H'(m)}{\sqrt{m^2-a^2}}\,da\right] \tag{47}$$

$$H'(a) = \frac{a-a_t}{a_t^2 a^3}\left(\left[\frac{2a_t}{a} + \frac{a-a_t}{m_t-a} + \frac{m_t(a-a_t)^2}{3a_t(m_t-a)^2}\right] g(a)\right.$$

$$+ \frac{a-a_t}{3a_t}\left(\frac{2a_t}{a} + \frac{m_t-a_t}{m_t-a}\right) h(a)$$

$$\left. - \frac{a-a_t}{2(m_t-a)\sqrt{m_t+m_t^2}\sqrt{a+a^2}}\left\{1 + \frac{a-a_t}{3a_t}\left[1 + \frac{m_t}{m_t-a} - \frac{1}{2(1+m_t)}\right]\right\}\right)$$

When $m_t=m$, $H'(m)$ becomes

$$\frac{m-a_t}{a_t^2 m^4 (1+m)}\left(\frac{a_t}{m} + \frac{(m-a_t)(1+2m)}{6a_t(1+m)}\left\{\frac{1}{2} + \frac{a_t}{m}\right.\right.$$

$$\left.\left. - \frac{(m-a_t)(1+2m)}{6m(1+m)}\left[\frac{1}{2} - 2m + \frac{1+2m}{m(1+m)}\right]\right\}\right)$$

III – WING WITH SUBSONIC TRAILING EDGE

The problem of satisfying the Kutta condition of zero pressure difference at the trailing edge and in the wake behind a highly swept wing is attacked in a manner similar to that used on the tip. The region behind the actual wing surface is divided into segments over which constant pressure increments may be assumed by extending the rays from the apex of the wing through points x_a, y_a on the trailing edge. This division of the wake region is indicated in figure 5, with a typical elementary sector a indicated by heavy lines.

ELEMENTARY SOLUTIONS FOR TRAILING EDGE

The element in this case differs from that in the preceding one in that the fixed side is no longer parallel to the stream but is inclined at the angle $\tan^{-1}\frac{m_t}{\beta}$ to it. The boundary conditions to be satisfied by the elementary solution are therefore (on the right-hand half of the wing)

$$\left.\begin{array}{llll} \left.\begin{array}{l}1.\\2.\end{array}\right\} & a \leq t_a \leq m_t & & u = \pm u_a \\ 3. & -1 < t_a < a & & u = 0 \\ 4. & m_t < t_a \leq +1 & & \bar{u} = 0 \end{array}\right\} \quad z = 0, \; \zeta = t_a$$

in addition to the general conditions 5 and 6 stated previously. The pertinent results of the preceding discussion will be assumed in order to shorten the discussion at this point.

The variable t_a is that defined by equation (18) and refers to the slope of rays originating at the apexes of the trailing-edge elements a. Along the trailing edge

$$y_a = \frac{m_t}{\beta}(x_a - c_o)$$

Since $a=\beta\frac{y_a}{x_a}$, we may solve for x_a and y_a as functions of a and the constants m_t and c_o:

$$\left.\begin{aligned} x_a &= \frac{m_t c_o}{m_t - a} \\ \beta y_a &= \frac{m_t c_o a}{m_t - a} \end{aligned}\right\} \tag{48}$$

Then

$$t_a = \frac{\beta y(m_t - a) - m_t c_o a}{x(m_t - a) - m_t c_o} \tag{49}$$

The solution for the oblique triangle satisfying the specified conditions is (in the plane of the wing)

$$u = \text{r.p.}\ \frac{u_a}{\pi}\cos^{-1}\frac{(1-a)(t_a - m_t) - (m_t - a)(1 - t_a)}{(1 - m_t)(t_a - a)} \tag{50}$$

Equation (21) will be seen to be the special case of equation (50) in which $m_t=0$, with the axes reversed because the relative positions of wing and element are now reversed. (See fig. 5.)

At this point a difficulty not encountered in the treatment of the tip must be considered. This difficulty is most apparent in connection with the element $a=0$, that is, the right triangle with apex at the trailing edge of the root section. The Mach cone from that point includes segments of both wings, so that condition 4 should properly be applied over part of the negative range of t_a as well as the positive. This could be done for $a=0$, but for any other small value of a, the area to be so treated would be nonconical with respect to the apex of the element a and therefore the boundary condition of zero downwash could not be fitted into the conical solution. Thus a deviation from the flat-plate boundary condition seems unavoidable. However, the error can be minimized by the choice of a solution for $a=0$ that does not introduce any downwash on the far wing.

The pressure at the root of the trailing edge is most simply disposed of by means of a symmetrical solution satisfying the following conditions:

$$-m_t \leq t_o \leq +m_t \qquad u = u_o \equiv u_\Delta(o)$$

$$\left.\begin{array}{l} -1 \leq t_o \leq -m_t \\ +m_t < t_o \leq +1 \end{array}\right\} \qquad \bar{u} = o$$

where t_o refers to rays emanating from the trailing edge of the root section $(a=0,\ x_a=c_o,\ y_a=0)$. The required solution is

$$u = \text{r.p. } \frac{u_o}{\pi} \cos^{-1} \frac{(1+m_t^2)t_o^2 - 2m_t^2}{(1-m_t^2)t_o^2} \tag{51}$$

From equation (10)

$$u_o = \frac{m\alpha V}{\beta E\left(\sqrt{1-m^2}\right)} \tag{52}$$

The major portion of the pressure in the wake will be canceled by solution (51); the remaining pressure may be canceled by the solutions given in equation (50) where u_a is, as before, $\frac{du_\Delta}{da}\,da$. The downwash induced on the far wing by the solutions should be, and has in fact been found to be, small, first, because du_Δ/da is small; and second, because as a and u_a increase, the portion of the far wing surface affected by the element a decreases in area and increases in distance from the element.

APPLICATION AT TRAILING EDGE

Pressure Distribution Near Trailing Edge

In determining the pressure at any point x,y on the wing surface it is first necessary to determine the most rearward element a_o that will influence the point x,y. As before the value of a_o is found by setting t_a (equation (49)) equal, in this case, to 1, so that

$$a_o = m_t \frac{\beta y + c_o - x}{\beta y + c_o m_t - x} \tag{53}$$

The velocity induced at any point within the Mach cone from the vertex of the trailing edge by superposition of the trailing-edge solutions discussed thus far is given by equation (51) plus the integral of equation (50) with respect to a, from $a=0$ to $a=a_o$. The total is to be subtracted from the basic distribution given by equation (10).

Total Induced Lift

If the Mach lines from the apex of the element a do not cross the leading edge of the wing, the conical element of area for the integration of the pressure to the tip is

$$dS = \frac{\beta m_t^2 s^2 (a_t - a)^2}{2a_t^2 (m_t - a)^2} \frac{dt_a}{t_a^2} \tag{54}$$

For the symmetrical solution, $a=0$ and

$$\frac{dS}{dt_o} = \frac{\beta s^2}{2t_o^2}$$

Integrating equation (51) over dS from $t_o=m_t$ to $t_o=1$ and multiplying by two to take account of the negative half of the wing gives, for the symmetrical solution,

$$\frac{\Delta L}{q\alpha} = \frac{-4s^2 m}{m_t E\left(\sqrt{1-m^2}\right)} \left[1 - \frac{2}{\pi} E\left(\sqrt{1-m_t^2}\right)\right] \tag{55}$$

When m_t is nearly 1.0, the expression in the brackets may be difficult to evaluate accurately. The following series expansion (from reference 8) is useful:

$$1 - \frac{2}{\pi} E\left(\sqrt{1-m_t^2}\right) = \frac{1-m_t}{2}\left[1 - R\left(\frac{1}{2^2} + \frac{1}{2^2 4^2} R^2 + \frac{1^2 3^2}{2^2 4^2 6^2} R^4 + \ldots\right)\right]$$

where $R = \frac{1-m_t}{1+m_t}$

The total correction to the lift induced by one of the oblique triangles in the wake is found by integrating equation (50) over dS as given by equation (54), from the trailing edge to $t_a=1$. The resulting formula is

$$(\Delta L)_a = \frac{-\rho V m_t^2 s^2 \beta (a_t-a)^2 u_a}{a_t^2 (m_t-a)^2}\left[\frac{1}{m_t} + \frac{1}{a}\left(\sqrt{\frac{(m_t-a)(1-a)}{m_t}} - 1\right)\right] \tag{56}$$

For the total induced correction to the lift due to the oblique elements, $(\Delta L)_a$ will have to be integrated over both wings, after substitution for u_a, and added to ΔL obtained from equation (55).

The integration has been done for both tapered and untapered wings and results in the following formulas:

For $m_t \neq m$,

$$\begin{aligned}\frac{\Delta L}{q\alpha} = \frac{-4m_t s^2}{a_t^2 (m_t^2-m^2) E\left(\sqrt{1-m^2}\right)} \Bigg(& \frac{m^2 (m_t-a_t)^2}{\sqrt{m_t^2-m^2}}\left[\cos^{-1}\frac{m^2-m_t a_t}{m(m_t-a_t)} - \cos^{-1}\frac{m}{m_t}\right] \\ & + (m^2-m_t a_t)\left\{2m-\sqrt{m^2-a_t^2}\left[1+\sqrt{\frac{m_t(1-a_t)}{m_t-a_t}}\right]\right\} - 2ma_t (m_t-a_t) \\ & + \sqrt{m_t(1-m)(m_t+m)}\left\{\frac{1}{2}\left[(m+a_t)^2\frac{m_t-m}{m_t+m} + (m-a_t)^2\frac{1+m}{1-m}\right]\left[F(\Phi_t, k)\right.\right. \\ & \left. - F(\Phi_o,k)\right] \\ & \left.\left. - \frac{1}{m_t^2-m^2}\left[(m^2-m_t a_t)^2 + 3m^2 (m_t-a_t)^2\right]\left[E(\Phi_t,k) - E(\Phi_o,k)\right]\right\}\Bigg)\end{aligned} \tag{57}$$

where

$$\Phi_t = \sin^{-1}\sqrt{\frac{(m_t+m)(m-a_t)}{2m(m_t-a_t)}},\ \Phi_o = \sin^{-1}\sqrt{\frac{m_t+m}{2m_t}} \text{ and } k = \sqrt{\frac{2m(1-m_t)}{(m_t+m)(1-m)}}$$

For $m_t = m$,

$$\frac{\Delta L}{q\alpha} = \frac{-2m^2s^2}{a_t^2E\left(\sqrt{1-m^2}\right)}\left\{\frac{4(m+a_t)}{3m^2}\sqrt{m^2-a_t^2} - \frac{m^2-a_t^2}{2m^2} + \frac{(m-a_t)^2}{8m^2}\left(\frac{5}{6} - \frac{1}{1-m}\right)\right.$$

$$-\frac{1}{4}\left[1 + \frac{3(m+a_t)}{m} - \frac{(m-a_t)(3-m)}{4m(1-m)}\right]\sqrt{\frac{(1-a_t)(m+a_t)}{m}}$$

$$-\frac{1+m}{2\sqrt{2}(1-m)}\left[1 - \frac{(m+a_t)^2(3-m)}{4m^2(1+m)} + \frac{(m-a_t)^2(5m-3)}{16m^2(1-m)}\right]$$

$$\times\left[\cosh^{-1}\frac{m(3-m)-(3m-1)a_t}{(1+m)(m-a_t)} - \cosh^{-1}\frac{3-m}{1+m}\right] \qquad (58)$$

Induced Moment

Any constantly loaded element of area $\frac{dS}{dt_a} dt_a$ from x_a, y_a has a moment arm about the Y-axis given by

$$\frac{\beta s}{a_t (m_t - a)} \left[(m_t - a_t) + \frac{2m_t (a_t - a)}{3t_a} \right] \tag{59}$$

For the symmetrical solution, $a=0$ and this expression reduces to

$$\beta s \left(\frac{m_t - a_t}{a_t m_t} + \frac{2}{3t_o} \right)$$

The moment induced on both halves of the wing by superposition of the symmetrical solution is, from equation (51)

$$\Delta M = \frac{2\rho V \beta s u_o}{\pi} \int_{m_t}^{1} \left(\frac{m_t - a_t}{a_t m_t} + \frac{2}{3t_o} \right) \cos^{-1} \frac{(1+m_t^2)t_o^2 - 2m_t^2}{(1-m_t^2)t_o^2} \frac{dS}{dt_o} dt_o$$

and

$$\frac{\Delta M}{q\alpha} = \frac{4m\beta s^3}{m_t^2 E\left(\sqrt{1-m^2}\right)} \left\{ \frac{m_t - a_t}{a_t} \left[1 - \frac{2}{\pi} E\left(\sqrt{1-m_t^2}\right) \right] + \frac{1-m_t^2}{6} \right\} \tag{60}$$

The moment induced by a single oblique element a is most conveniently written in two parts, corresponding to the two terms of equation (59):

$$(\Delta M)_{a,1} = \frac{-\beta s (m_t - a_t)}{a_t (m_t - a)} (\Delta L)_a \tag{61(a)}$$

and

$$(\Delta M)_{a,2} = \frac{4\rho V\beta s m_t (a_t-a)u_a}{3a_t(m_t-a)\pi} \int_{m_t}^{1} \frac{1}{t_a} \cos^{-1} \frac{(1-a)(t_a-m_t)-(m_t-a)(1-t_a)}{(1-m_t)(t_a-a)} \frac{dS}{dt_a} dt_a$$

$$= \frac{\rho V\beta^2 m_t^3 s^3 (a_t-a)^3 u_a}{3a_t^3 (m_t-a)^3} \left[\frac{1}{m_t^2} - \frac{1}{a^2} + \frac{1}{2a}\left(\frac{1}{m_t} + \frac{2}{a} + 1\right)\sqrt{\frac{(m_t-a)(1-a)}{m_t}}\ \right] \tag{61(b)}$$

The total induced change in pitching moment due to the cancellation of pressure at the trailing edge is then

$$\Delta M + \int_0^a \left[(\Delta M)_{a,1} + (\Delta M)_{a,2}\right] da \tag{62}$$

where the first term is the effect of the symmetrical solution (equation (60)).

The integral forming the second part of equation (62) has not been evaluated analytically. For numerical integration, when m_t is close to 1.0, the following series expansions are suggested.

In $(\Delta L)_a$,

$$\left[\frac{1}{m_t} + \frac{1}{a}\left(\sqrt{\frac{(m_t-a)(1-a)}{m_t}} - 1\right)\right] = \frac{1-m_t}{2m_t}\left(1 - \frac{1}{4}Q - \frac{1.3}{4.6}Q^2 - \frac{1.3.5}{4.6.8}Q^3 \ldots\right)$$

where $Q = \dfrac{a(1-m_t)}{m_t(1-a)}$

In $(\Delta M)_{a,2}$, the corresponding bracketed expression equals

$$\frac{1-m_t}{4m_t^2}\left[3+m_t-\frac{Q}{2}\left(\frac{1+m_t}{2}+\frac{m_t}{a}\right)\left(1+\frac{1.3}{6}Q+\frac{1.3.5}{6.8}Q^2+\ldots\right)\right]$$

SUCCESSIVE STEPS FOR COMPLETE SOLUTION

Application of the procedure described so far will result in cancellation of the pressure along the forward part of the tip and along the inboard portion of the trailing edge. Over that part of the tip that lies within the Mach cone from the apex of the trailing edge, however, the method of removal of the trailing-edge pressures has induced an extraneous negative pressure, for which an additional correction will have to be made. Similarly, the tip solutions have introduced negative pressure along that portion of the trailing edge falling within the Mach cones from the tip. There is also a complex distribution of pressure in the wake and in the stream beyond the wing tips.

The lift restored to the wing as a result of the cancellation of these erroneous negative pressures will be referred to as secondary corrections to the basic lift.

Figure 6 shows the regions near the wing tip affected by these secondary corrections. The errors introduced by the primary corrections occur along and behind the trailing edge from $t_m = -1$ to the tip, and along and outboard of the tip from $t_o = 1$ to the trailing edge. Since in general very little of the wing surface lies within the Mach cones (shown shaded in fig. 6) from these regions, and the actual flow pattern of the viscous fluid would differ considerably from what we would calculate, it is questionable how far one should go in attempting to correct for the residual pressures there. The exact procedure will be outlined here, and may be feasible when a mechanical computer is available to perform the integrations numerically. A method for approximating the pressure distribution, lift and moment will be suggested in a following section.

It should first be noted that the distribution of residual lift in the field is no longer conical with respect to any one point, but is composed of an infinite number of superimposed conical fields originating at various points along the trailing edge and tip. In order to cancel these pressures accurately, it is necessary to go back to the application of the elementary solutions at the wing tip and trailing edge and cancel the extraneous pressures introduced by each one. This will involve, for each element, the integration of an infinite number of elementary solutions of the other type; that is, trailing-edge-induced pressures will be canceled by the application of tip solutions, and vice versa.

Consider an element at the trailing edge (fig. 7) with uniform pressure u_a, giving rise to a pressure field defined by equation (50) in terms of t_a, the angular displacement around x_a,y_a on the trailing edge. The velocity

$$(\Delta u)_a = \frac{u_a}{\pi} \cos^{-1} \frac{(1-a)(t_a-m_t)-(m_t-a)(1-t_a)}{(1-m_t)(t_a-a)} \tag{63}$$

with

$$t_a = \beta\frac{y_b-y_a}{x_b-x_a} \tag{64}$$

induced at points x_b,y_b on the wing tip by the individual element a may then be removed by superposing triangular elements at the tip formed by extensions of the rays t_a from x_a,y_a on the trailing edge. Such tip elements in turn induce velocities within their Mach cones which may be expressed in terms of the angular displacement around the apexes of the element x_b,y_b. This displacement may be denoted by the parameter

$$t_b = \beta\frac{y-y_b}{x-x_b} \tag{65}$$

which is a function of a and t_a. Then, from equation (21), the effect of a single tip element bounded by the ray t_a is proportional to

$$\frac{1}{\pi} \cos^{-1} \frac{t_a+t_b+2t_at_b}{t_b-t_a} \tag{66}$$

Since $(\Delta u)_a = 0$ at $t_a = 1$, the total velocity induced at a point x,y on the wing by removal of all tip pressures resulting from the wake element a will be

$$-\frac{1}{\pi}\int_{t_a=1}^{t_b=-1} \frac{d(\Delta u)_a}{dt_a} \cos^{-1} \frac{t_a+t_b+2t_at_b}{t_b-t_a}\, dt_a \tag{67}$$

The derivative $d(\Delta u)_a/dt_a$ is infinite at the lower limit. In order to perform the indicated integration graphically it is preferable to write $\frac{d(\Delta u)_a}{dt_a} dt_a = d(\Delta u)_a$ and integrate by plotting the inverse cosine function against $(\Delta u)_a$ for the indicated range of values of t_a.

The foregoing procedure must in turn be followed for values of a from $a=0$ back to that value of a for which $t_b\ (t_a,x,y) = -1$ when $t_a=1$ (i.e., for which the point x,y lies on the reflected Mach line from x_a,y_a), and the results integrated with respect to a.

The integrated expressions, (29) for the lift and (33) for the moment due to each tip element, apply to the elements from x_b,y_b if a is replaced in the functions g and h by t_a and elsewhere by $a(x_b,y_b) = \beta y_b/x_b$. In the coefficient, $-d(\Delta u)_a/dt_a$ will replace u_a as the constant velocity over the section. For the total increment in lift and moment caused by correcting for a single wake element a, $(\Delta L)_b$ and $(\Delta M)_b$ obtained in this way would have to be integrated with respect to t_a from $t_a = m_t$ to 1. Again, since $d(\Delta u)_a/dt_a$ is infinite at the limits, it is preferable to integrate with respect to $(\Delta u)_a$. The results would have to be integrated again with respect to a, from $a=0$ to $a=a_t$.

In canceling the tip pressures introduced by the symmetrical solution (equation (51)), the same procedure would be used with $(\Delta u)_a$ replaced by

$$\frac{u_o}{\pi}\cos^{-1}\frac{(1+m_t^2)t_o^2-2m_t^2}{(1-m_t^2)t_o^2}$$

and $t_a = t_o$, $x_a = c_o$ and $y_a = 0$. Integration with respect to a is, of course, not required.

A similar procedure to that just outlined would be followed in canceling the extraneous pressures introduced at the trailing edge by the original application of canceling solutions at the tip. In this case, however, the presence in the basic solution of infinite velocities at the leading edge leads to considerable difficulty in obtaining a satisfactory method for numerical integrations. Since the results presented in figure 4 suggest that the wing tip should be raked inward for efficient design, it does not appear worthwhile to develop this part of the solution in any further detail. The approximate method for treating the extraneous pressures at the tip presented in the following section is considered satisfactory for practical investigation of the conventional wing tip.

Where the Mach number is very low (close to 1.0), it might be necessary to carry the foregoing procedure through another step, canceling the pressures introduced at the opposite edge by the new solutions in t_b. It is clear that the process outlined is a converging one, since (1) the functions giving the pressure distribution are everywhere smaller in value than the pressure being removed, and (2) the area affected by each successive step is smaller than the preceding. The approximations suggested in the following section may be made at any later step in the calculations.

METHODS FOR APPROXIMATE SOLUTION

The following method of approximating the secondary corrections will result in cancellation of the extraneous pressures at the trailing edge and along the tip of the wing, but will not take account of the exact variation of pressure in the stream. It will be apparent however, that the residual errors will be small, particularly in their effect on the wing.

Correction for Tip-Induced Pressures

The assumption will be made that since the pressures induced by the tip are in the main due to cancellation of the infinite pressure at the leading edge, the extraneous pressure field introduced by the tip solutions will be very nearly canceled by a pressure field conical from the tip of the leading edge (fig. 8), provided the pressures are made to cancel exactly along the trailing edge. In the conical pressure field to be superimposed, the velocities are held constant at Δu (equation (37)) along the rays t_m projected from the tip of the leading edge back into the wake. To effect this approximate cancellation, values of Δu are calculated from equation (37) for points x_b,y_b along the trailing edge. The corresponding values of t_m are

$$t_m = \beta\frac{y_b-s}{x_b - \frac{\beta s}{m}} \tag{68}$$

Let the particular point at which the line $t_m = -1$ intersects the trailing edge be designated by x^*, y^* and other symbols referring to that point be similarly starred. Then the velocity induced at any point x,y on the wing by removal of Δu along the trailing edge will be (from equation (50))

$$\frac{-\Delta u^*}{\pi} \cos^{-1} \frac{2(t^*-m_t)-(m_t+1)(1-t^*)}{(1-m_t)(t^*+1)}$$

$$-\frac{1}{\pi}\int_{-1}^{t_b=1} \frac{d\Delta u}{dt_m} \cos^{-1} \frac{(1-t_m)(t_b-m_t)-(m_t-t_m)(1-t_b)}{(1-m_t)(t_b-t_m)}\, dt_m \qquad (69)$$

where Δu^* is given by equation (38), t^* is the parameter $\beta\frac{y-y^*}{x-x^*}$ and t_b is the previously defined parameter (equation (65)) measuring displacement around other points $x_b y_b$ on the trailing edge. The upper limit of integration is a function of the point x,y at which the pressure is being determined.

The derivative $d\Delta u/dt_m$ would have to be determined numerically or graphically from a plot of the calculated values of Δu against t_m. In order to avoid this procedure, it is preferable to rewrite equation (69) as

$$\frac{-\Delta u^*}{\pi} \cos^{-1} \frac{2(t^*-m_t)-(m_t+1)(1-t^*)}{(1-m_t)(t^*+1)}$$

$$-\frac{1}{\pi}\int_{\Delta u^*}^{t_b=1} \cos^{-1} \frac{(1-t_m)(t_b-m_t)-(m_t-t_m)(1-t_b)}{(1-m_t)(t_b-t_m)}\, d\Delta u \qquad (70)$$

and integrate by plotting the inverse cosine function against Δu.

For the total change in lift and moment resulting from this correction, equations (56) and (61(a) and (b)) would be used, with the following substitutions:

for u_a, Δu^* and $d\Delta u/dt_m$

for a, outside the brackets, $\beta y^*/x^*$ and $\beta y_b/x_b$

for a, inside the brackets or in Q, -1 and t_m

The total secondary tip correction to the lift (over both halves of the wing) would then be (from equation (56))

$$\Delta_2 L = \frac{-2\rho V m_t^2 s^2 \beta}{a_t^2}\left\{\Delta u^*\left(\frac{a_t x^* - \beta y^*}{m_t x^* - \beta y^*}\right)^2\left(\sqrt{\frac{1+m_t}{m_t}} - \sqrt{2}\right)\sqrt{\frac{1+m_t}{m_t}}\right.$$

$$\left. + \int_{-1}^{0}\left(\frac{d\Delta u}{dt_m}\right)\left(\frac{a_t x_b - \beta y_b}{m_t x_b - \beta y_b}\right)^2\left[\frac{1}{m_t} + \frac{1}{t_m}\left(\sqrt{\frac{(m_t - t_m)(1-t_m)}{m_t}} - 1\right)\right] dt_m\right\} \tag{71}$$

which should again be integrated by rewriting $(d\Delta u/dt_m)\ dt_m$ as $d\Delta u$. The corresponding correction to the moment can be written similarly from equation (61). The expressions in the brackets become indeterminate when $t_m = 0$. Their limiting values may be obtained directly from the series expansions in Q, since $Q(o) = 0$.

Correction for Trailing-Edge Induced Pressures

The residual pressure at the tip induced by the application of the trailing-edge solutions has a somewhat different distribution from that induced at the trailing edge by the tip solution. A procedure corresponding to that just described – that is, assuming that the pressures induced by the oblique solutions, as well as by the symmetrical solution, all originate at the center of the trailing edge – is not so readily justified as in the preceding case. However, the pressures induced by the oblique solutions are quite small, and their secondary effect, particularly on the total lift, might in fact be neglected, if it were not comparatively simple to take them into account with this assumption.

The approximation is made by substituting in the previously given equation (67) for the secondary trailing-edge correction to the local velocity at any point x,y, the value of

$$t_o = \frac{\beta s}{x_b - c_o}$$

for t_a. The approximate induced velocity correction is therefore

$$-\frac{1}{\pi}\int_{t_o=1}^{t_b=-1} \cos^{-1}\frac{t_o + t_b + 2t_o t_b}{t_b - t_o}\, d\Delta u \tag{72}$$

where Δu is the total (negative) velocity remaining at the points x_b, s along the tip, found by summing the values from equation (51) and the integral of equation (50).

Equations (29) and (33) are applicable in calculating the total correction to the lift and moment, with the following substitutions:

for u_a, $d\Delta u/dt_o$

for a, except in $g(a)$ and $h(a)$, $\beta s/x_b$

for $g(a)$ and $h(a)$, $g(t_o)$ and $h(t_o)$

Thus the total secondary trailing-edge change in lift (both halves of the wing) would be

$$\Delta_2 L = \frac{2\rho V m_t^2}{\beta a_t^2} \int_{t_o=m_t}^{t_o=1} (a_t x_b - \beta s)^2 \; g(t_o) \; d\Delta u(t_o) \tag{73}$$

and the corresponding correction to the moment can be written similarly from equation (33).

Further Simplification of Lift and Moment Calculations and Summary of Formulas

Effect of tip.- The primary corrections to the lift and moment due to reducing the pressures at the tip to zero are obtained without difficulty from equations (43) through (47). Calculation of the induced pressures at the trailing edge in order to make the secondary correction is, however, somewhat tedious. These pressures vary very little over the Mach cone from the tip, and the variation is nearly linear. This calculation and, in addition, the integration in equation (71) can therefore be eliminated in many cases without any significant loss in accuracy by considering Δu to be constant at some intermediate value between Δu^* and $-u_\Delta(a_t)$, the value at the tip. The secondary correction is then simply

$$\Delta_2 L = \frac{-2\rho V m_t^2 s^2 \beta}{a_t^2} \left(\frac{a_t x^* - \beta y^*}{m_t x^* - \beta y^*}\right)^2 \sqrt{\frac{1+m_t}{m_t}} \left(\sqrt{\frac{1+m_t}{m_t}} - \sqrt{2}\right) \Delta u \tag{74}$$

where Δu may be the average of Δu^* (equation (30)) and $-u_\Delta(a_t)$ (equation (10)).

Effect of trailing edge.- The major part of the primary trailing-edge correction is accomplished by the application of the symmetrical solution and, in many cases, the integration of equations (56) and (59) can be omitted in calculating the total lift and moment. Then the primary effect is given by equation (55). Calculations of the secondary effect may be simplified very greatly by ignoring the effect of the oblique elements, and using Δu as calculated directly from equation (51) in equation (73) and in the corresponding equation for the moment.

WING WITH CROSS-STREAM TIPS

The fact, mentioned earlier in this paper, that a drastic reduction in lift takes place behind the Mach cone from the tip of the highly swept wing suggests immediately the consideration of the effect of cutting off the tip areas along a line normal to the stream. The resulting plan form (fig. 9) presents obvious structural advantages and may also be expected to approach the triangular wing in its longitudinal stability characteristics.

In calculating the pressure distribution over such a wing, only the trailing-edge solutions apply. For the total lift and moment, new integrations must be performed, since the elements of area are now different. The characteristic ray a_t is now replaced by one through the inboard end of the cross-stream tip (a_1 in fig. 9). For a less than this value,

$$\frac{dS}{da} = \frac{\beta s^2}{2m^2}$$

and

$$\frac{dS}{dt_a} = \frac{1}{2\beta}\left(\frac{\beta s}{m} - x_a\right)^2$$

x_a being defined by equation (48). It is convenient to replace c_o by the following expression

$$c_o = \frac{\beta s}{m m_t}\left(m_t - a_1\right)$$

Then the basic lift is

$$\frac{L_o}{q\alpha} = \frac{4s^2}{E\left(\sqrt{1-m^2}\right)}\left[\frac{(m_t-a_1)^2}{(m_t^2-m^2)}\left\{\frac{m_t}{\sqrt{m_t^2-m^2}}\left[\cos^{-1}\frac{m^2-a_1m_t}{m(m_t-a_1)} - \cos^{-1}\frac{m}{m_t}\right]\right.\right.$$

$$\left.\left. - \frac{\sqrt{m^2-a_1^2}}{m_t-a_1} + \frac{m}{m_t}\right\} + \cos^{-1}\frac{a_1}{m}\right] \tag{75}$$

When $m_t = m$,

$$\frac{L_o}{q\alpha} = \frac{4s^2}{E\left(\sqrt{1-m^2}\right)}\left\{\cos^{-1}\frac{a_1}{m} + \frac{1}{3m^2}\left[(2m-a_1)\sqrt{m^2-a_1^2} - 2(m-a_1)^2\right]\right\} \tag{76}$$

The decrement of lift due to application of the symmetrical solution is (provided, as before, that the trailing-edge Mach lines do not intersect the leading edge)

$$\frac{\Delta L}{q\alpha} = \frac{-4a_1^2s^2}{mm_tE\left(\sqrt{1-m^2}\right)}\left[\frac{2}{\pi}K\left(\sqrt{1-m_t^2}\right) - 1\right] \tag{77}$$

Except for very high aspect-ratio wings, the trailing edge of this configuration will lie entirely in that range over which du_Δ/da is negligible compared to u_o. For this reason, the integrations necessary to find $(\Delta L)_a$ for this case have not been performed. They are, however, elementary in nature and can be readily carried through by the designer considering a wing of this type.

The moment of the basic solution about the apex of the wing is

$$\frac{M_o}{q\alpha} = \frac{4\beta s^3}{3mE\left(\sqrt{1-m^2}\right)}\left[\frac{(m_t-a_1)^2}{m_t^2-m^2}\left\{\frac{(2m_t^2+m^2)(m_t-a_1)}{(m_t^2-m^2)^{3/2}}\left[\cos^{-1}\frac{m^2-a_1m_t}{m(m_t-a_1)}\right.\right.\right.$$

$$\left. - \cos^{-1}\frac{m}{m_t}\right] - \left(\frac{1}{m_t-a_1} + \frac{3m_t}{m_t^2-m^2}\right)\sqrt{m^2-a_1^2}$$

$$\left.\left. + \frac{m(m_t-a_1)}{m_t^2-m^2} + \frac{m(m_t-a_1)}{m_t^2}\right\} + 2\cos^{-1}\frac{a_1}{m}\right] \tag{78}$$

When $m_t = m$,

$$\frac{M_o}{q\alpha} = \frac{8\beta s^3}{3mE\left(\sqrt{1-m^2}\right)} \left\{\frac{1}{5m}\sqrt{m^2-a_1^2}\left[1 + \frac{2(m-a_1)}{3m}\left(1 + \frac{m-a_1}{m}\right)\right] - \frac{7(m-a_1)^3}{15m^3} + \cos^{-1}\frac{a_1}{m}\right\} \tag{79}$$

The decrement in moment due to the removal of u_o at the trailing edge is simply $c_o + \frac{2}{3}\left(\frac{\beta s}{m} - c_o\right)$ times the decrement in lift, or

$$\Delta M = \frac{\beta s\,(3m_t-a_1)}{3mm_t}\,\Delta L \tag{80}$$

where ΔL is given by equation (77).

Computations have been made for a wing of this type and will be presented in a later section.

OTHER APPLICATIONS OF THE OBLIQUE SOLUTIONS

Low Aspect Ratio Wings

Variations of the method of this paper could be used to obtain the pressure distributions for a variety of rectilinear plan forms other than the conventional swept wing. For example, the reverse delta wing (fig. 10) with trailing edges falling within the Mach cones from the tips can be treated by using as the basic solution that for a rectangular wing with raked tips (given by Hayes, reference 11, and others), the deficiencies from the two-dimensional pressure being additive where the Mach cones overlap. The oblique solutions of the present report would be applied to cancel the pressure deficiency introduced along the rear of each trailing edge by the simple conical field from the opposite tip. Other low-aspect-ratio wings can be treated similarly, provided cancellation of pressure ahead of a leading edge is not involved.

Reverse Taper

Extension of the procedure to wings with reverse taper is obvious. In this case the wing with subsonic trailing edge may have either subsonic or supersonic leading edge. (The case of swept-forward leading edges requires separate consideration.) Starting with the appropriate basic solution, the subsequent procedure and formulas of the present report will apply without modification.

IV – NUMERICAL EXAMPLES

The method has been applied to two wings with 63° sweepback of the leading edge, one with considerable taper and one with constant chord, at a Mach number of 1.5. The wings are shown in figures 11 and 12. From the dimensions given, the following quantities, required for the computations, can be calculated:

Parameter	Tapered	Untapered
β	1.118	1.118
m	.570	.570
$E\left(\sqrt{1-m^2}\right)$	1.256	1.256
m_t	.906	.570
a_t	.528	.358
S	26 sq ft	58 sq ft

It should be noted that the dimensions and calculated values omit the tip fairings, which are formed by rotating the (symmetrical) tip section about its chord line. The vortex sheet with which a lifting surface may be replaced for linear potential flow calculations, would have a similar rounding off outboard of the last vortex without adding any lift.

The basic velocity distribution (equation (10)), which depends only on m, is the same for both wings. Corrections to this distribution, taking place within the Mach cones from the trailing edge and tip, cover so small an area of the tapered wing (fig. 11), the trailing edge of which almost coincides with the Mach lines, that the variation within that area is not of much interest. The effect on the total lift and moment will be given later. The pressure distribution over the constant chord wing will be discussed first in some detail.

PRESSURE DISTRIBUTION OVER SWEPT UNTAPERED WING

In practice, it is convenient to compute all velocities in terms of u_o, the basic perturbation velocity along the X-axis, in order to avoid dimensionality and numerous multiplications by a constant factor. All but the final figures will therefore show the velocities in ratio form.

The decrement in velocity resulting from removal of constant pressure across the wake (symmetrical solution, equation (51)) is plotted against t_o in figure 13. This component is of course dependent only on the sweep of the trailing edge relative to the Mach lines and would apply equally well to wings of any span and taper having the same trailing-edge sweep angle.

The decrement in velocity due to removal of the remaining pressures at the trailing edge was found at various stations spanwise and chordwise within the Mach cone from the center of the trailing edge and added to the decrement shown for the corresponding value of t_o in figure 13. Evaluation of this second component involved the graphical integration, for each point, of equation (50) with respect to a. Figure 14 shows some typical plots of this function. The value of a at which each curve falls to zero is a_o (equation (53)) for the specified value of x and y. The velocities obtained in this way are plotted spanwise for two values of x in figure 15.

The spanwise distribution of velocity decrement within the Mach cone from the tip, calculated by equation (37), is plotted in figure 16.

Figures 13, 15, and 16 represent only the primary corrections. In figures 17 (a) to (d) these corrections have been applied to the basic velocity distribution along the chord at each of four spanwise stations, and the results converted to terms of $\Delta p/q\alpha$ by the factor $4u_o/\alpha V = 4m/\beta E(\sqrt{1-m^2})$. The magnitude of the extraneous negative pressures at the trailing edge remaining after the first approximation can be seen in the last two of these figures. Corrections have been added to eliminate those errors due to the tip solution by the approximate method (equation (70)) of the preceding section. The errors introduced by the symmetrical solution have been eliminated without resort to approximate methods. The errors induced by the wake gradient solutions were neglected because of their location far back on the tip and their relatively small values. The effects of these secondary corrections are shown in figures 17 (c) and (d). For the final curves the pressure distribution has been arbitrarily faired to zero, taking cognizance only of the points at which succeeding corrections enter. It can be seen by drawing another set of reflected Mach lines at the tip of figure 12 that the area affected by this approximation is very small and is in that region where viscous effects would tend most to invalidate any results calculated by the present simplified theory. A three-dimensional view of the final pressure distribution is shown in figure 18.

LIFT AND MOMENT COEFFICIENTS

The various components of the lift, with the equations from which they were obtained, are given in table I for both the tapered and constant-chord wing. The last correction was made in two parts, one for the symmetrical solution and one for the effect of the oblique solutions, assuming the latter to be conical with respect to the apex of the trailing edge, in order to estimate the magnitude of the error involved in such an assumption. The value given represents the upper limit of the possible error. In the secondary correction for the tip effect (correction at the trailing edge) 90 percent of the effect was obtained from the first term of equation (71), only 10 percent being contributed by the integral.

The value of C_{L_α} obtainable by discarding the trailing-tip regions of the constant-chord wing was also estimated, using equations (76) and (77). The modified wing had the same span as the original wing, but its area was decreased to 40.8 square feet and its aspect ratio increased from 1.725 to 2.45 by removal of the trailing tips. As a result the value of C_{L_α} increased from 2.01 to 2.63, approaching that for the highly tapered wing.

(It may be remarked here that the pressure distribution for the modified wing is identical with that over the original wing (fig. 17) up to the point at which the tips are cut off, where the pressure drops discontinuously to zero.)

The components of pitching moment for the three wings are presented in table II. Second-approximation corrections were not made where the corresponding component of lift was negligible.

DRAG DUE TO LIFT

With the value of C_{L_α} calculated, it is possible to find immediately the drag due to lift for the specified wings. As pointed out in reference 12, where similar calculations are made for a more limited class of wings, the flat wing at an angle of attack is subject to a drag which is merely the component of the lift in the flight direction, or α times the lift, and to a thrust which is due to the infinite suction force at the leading edge. If, as in the examples calculated, the Mach lines from the trailing edge do not cross the leading edge, the thrust on the leading edge is exactly that on the triangular wing of equal span at the same angle of attack. From reference 12, this thrust, in coefficient form, is

$$\frac{\beta\sqrt{1-m^2}}{4\pi m} {C_{L_\Delta}}^2$$

where C_{L_Δ} is the lift coefficient of the triangular wing. From reference 6, C_{L_α} of the triangular wing with $m<1$ is $\frac{2\pi m}{\beta E\left(\sqrt{1-m^2}\right)}$. Using this value and the area of the triangular wing $\beta s^2/m$, the thrust force can be expressed in terms of the sweep of the trapezoidal wing as follows:

$$T = \frac{\pi s^2\sqrt{1-m^2}}{\left[E\left(\sqrt{1-m^2}\right)\right]^2} q\alpha^2$$

The total drag for the swept wing is therefore

$$q\alpha^2 C_{L_\alpha} S - T = q\alpha^2 \left\{ C_{L_\alpha} S - \frac{\pi s^2\sqrt{1-m^2}}{\left[E\left(\sqrt{1-m^2}\right)\right]^2} \right\}$$

and the drag coefficient may be written

$$C_D = \left\{C_{L_\alpha} - \frac{\pi\sqrt{1-m^2}}{\left[E\left(\sqrt{1-m^2}\right)\right]^2} \left(\frac{s^2}{S}\right) \right\}\alpha^2 \qquad (81)$$

or

$$\frac{C_D}{{C_L}^2} = \frac{1}{{C_{L_\alpha}}^2} \left\{C_{L_\alpha} - \frac{\pi\sqrt{1-m^2}}{\left[E\left(\sqrt{1-m^2}\right)\right]^2} \left(\frac{s^2}{S}\right)\right\} \qquad (82)$$

For the three wings described, the results may be summarized as follows:

Wing	Aspect ratio	C_{L_α}	$C_D/C_L{}^2$
Tapered wing	3.85	3.09	0.160
Untapered wing	1.73	2.01	.323
Modified untapered wing	2.45	2.63	.253

CONCLUDING REMARKS

The calculation of the theoretical load over a swept-back wing of moderate aspect ratio at a Mach number not too close to one appears to be feasible by the method described in this paper. The total lift can be approximated within 5 percent for a considerable range of practical plan forms with only the first step of the successive approximations outlined. In the cases calculated, the major effects were obtained with only one graphical integration for each wing (that for the primary tip effect). The moment is not as readily approximated; some secondary corrections will generally prove necessary.

Although in the examples calculated the oblique solutions contributed only about 1 percent to the final lift and 2 percent to the moment, they are expected to prove useful for other cases, particularly for low-aspect-ratio wings of other than rectangular plan form.

At Mach numbers close to 1.0, the method is no longer practical especially for predicting pressure distributions. It appears that some other line of approach is required for this case.

The results of the calculations emphasize the indications of the theory that the determining factor in the aerodynamic performance of wings, as far as differences in plan form are concerned, is the disposition of area relative to the Mach lines. The apparent beneficial effect of taper and increased aspect ratio, for example, is largely the result of reducing the percentage of area within the Mach cones from the tip and trailing edge. The negative lift calculated for the tip areas, in particular, may be expected to have a serious effect on the longitudinal stability variation with Mach number. With the proper thickness distribution, the wing with cross-stream tips may provide the solution to this problem.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif.

APPENDIX A

Summary of Symbols

x,y,z — cartesian coordinates in the stream direction, across the stream, and in the vertical direction, respectively

V — free-stream velocity

u,v,w — perturbation velocities in the stream direction, across the stream, and in the vertical direction, respectively

M — free-stream Mach number

β — $\sqrt{M^2-1}$

ρ — density of air

q — dynamic pressure $\left(\frac{1}{2}\rho V^2\right)$

Δp — pressure difference between upper and lower surfaces, or local lift

c_o — root chord

s — semispan

S — wing area

α — angle of attack, radians

Λ — angle of sweep of the leading edge

m — $\dfrac{\text{slope of leading edge}}{\text{slope of Mach lines}} = \beta \cot \Lambda$

m_t — $\dfrac{\text{slope of trailing edge}}{\text{slope of Mach lines}}$

a — $\dfrac{\text{slope of any ray through origin}}{\text{slope of Mach lines}} = \dfrac{\beta y}{x}$

Also used to designate a ray through the origin or a constant-load element having such a ray as one side, the other side being a boundary of the wing surface. Used as a subscript to designate any quantity pertaining to such an element.

a_o value of a corresponding to most rearward element affecting a specified point x,y. On tip, $a_o = \frac{\beta s}{x+\beta(y-s)}$

On trailing edge, $a_o = m_t \frac{\beta y+c_o-x}{\beta y+c_o m_t-x}$

a_t $\frac{\text{slope of ray through trailing-edge tip}}{\text{slope of Mach lines}} = \frac{\beta s}{c_o+\beta s/m_t}$

x_a,y_a coordinates of apex of any constant-load element a

x_b,y_b coordinates of apex of elements used in secondary corrections

x^*,y^* coordinates of the point at which the Mach line from the tip crosses the trailing edge

t_a $\frac{\text{slope of ray through apex of element a}}{\text{slope of Mach line}} = \beta\frac{y-y_a}{x-x_a}$ for any point x,y

t_b $\frac{\text{slope of ray through } x_b,y_b}{\text{slope of Mach lines}} = \beta\frac{y-y_b}{x-x_b}$

t_o $\frac{\text{slope of ray through trailing-edge apex}}{\text{slope of Mach lines}} = \beta\frac{y}{x-c_o}$

t_m $\frac{\text{slope of ray from leading-edge tip}}{\text{slope of Mach line}} = \beta\frac{y-s}{x-\frac{\beta s}{m}}$

t^* $\frac{\text{slope of ray through } x^*,y^*}{\text{slope of Mach lines}} = \beta\frac{y-y^*}{x-x^*}$

u_Δ basic perturbation velocity as given by solution for triangular wing

u_1 value of u_Δ for wing with supersonic leading edge, when $a \geqq 1$

u_o value of u_Δ for wing with subsonic leading edge when $a=0$ (center line of wing)

u_a incremental velocity on any constant-load element a

Δu correction to the basic incremental velocity at any point

$(\Delta u)_a$ correction to the basic incremental velocity induced by a single element a

Δu^*	Decrement in velocity u induced at x^*,y^* by cancellation of pressures at the tip
L_o	basic lift for entire wing
M_o	basic moment
$(\Delta L)_a$	correction to total lift due to application of one constant-load element
ΔL	correction to the basic lift
Φ	complex velocity potential function
φ	real part of Φ
$\overline{\varphi}$	imaginary part of Φ
U	complex velocity function of which u is the real part
$\overline{u}$	imaginary part of U
ϵ	$=\dfrac{y+iz}{x+\sqrt{x^2-\beta^2y^2-\beta z^2}}$, argument for solutions of the supersonic flow equation
ζ	$=\dfrac{2\beta\epsilon}{1+\beta^2\epsilon^2}$
$g(a)$	$=\dfrac{1}{m_t-a}\left(\sqrt{\dfrac{a+a^2}{m_t+m_t^2}}-\dfrac{a}{m}\right)$
$h(a)$	$=\left(1+\dfrac{m_t}{m_t-a}\right)g(a)-\dfrac{1}{2(1+m_t)(m_t-a)}\sqrt{\dfrac{a+a^2}{m_t+m_t^2}}$
Q	$=\dfrac{a(1+m_t)}{m_t(1-a)}$
E	elliptic integral of the second kind
F	elliptic integral of the first kind
K	complete elliptic integral of the first kind

APPENDIX B

Integration for Loss of Lift at the Tip of the Highly Swept Wing

From equations (42), (44), and (10)

$$\frac{\Delta L}{q\alpha} = -\frac{4m_t^2 m^2 s^2}{E\left(\sqrt{1-m^2}\right)} \int_{a_t}^{m} \frac{G'(a)}{\sqrt{m^2-a^2}}\, da \tag{A1}$$

where

$$G'(a) = \frac{a-a_t}{a_t^2 a^2 (m_t-a)} \left[\left(\frac{a_t}{a} + \frac{m_t-a_t}{m_t-a} \right) \left(\sqrt{\frac{a+a^2}{m_t+m_t^2}} - \frac{a}{m_t} \right) - \frac{a-a_t}{2\sqrt{m_t+m_t^2}\sqrt{a+a^2}} \right] \tag{A2}$$

The terms in $G'(a)$ are of two types; namely, those that contain $\sqrt{a+a^2}$ and those that do not. The former combine with the radical $\sqrt{m^2-a^2}$ in (A1), to form elliptic integrals of the first, second, and third kinds. The latter give rise to terms in equation (A1) which are immediately integrable by elementary means. It is convenient, therefore, to consider the integral in two parts, writing

$$-\int_{a_t}^{m} \frac{G'(a)}{\sqrt{m^2-a^2}}\, da = I_1 + I_2$$

where I_1 is that part of the integral not requiring elliptic integrals.

Then

$$I_1 = \int_{a_t}^{m} \frac{a-a_t}{a_t^2 m_t a (m_t-a)} \left(\frac{a_t}{a} + \frac{m_t-a_t}{m_t-a} \right) \frac{da}{\sqrt{m^2-a^2}}$$

$$= \frac{1}{a_t^2 m_t (m_t^2-m^2)} \left[\frac{(m_t-a_t)^2}{\sqrt{m_t^2-m^2}} \cos^{-1} \frac{m_t a_t - m^2}{m(m_t-a_t)} + \frac{m^2-m_t a_t}{m^2} \sqrt{m^2-a_t^2} \right] \tag{A3}$$

The remaining terms, involving $\sqrt{a+a^2}$ and $\sqrt{m^2-a^2}$, are integrated by means of the substitution[3]

$$u = sn^{-1} \left[\sqrt{\frac{m-a}{m(a+1)}} , \sqrt{\frac{1-m}{2}}\right]$$

so that

$$\left.\begin{aligned} sn\ u &= \sqrt{\frac{m-a}{m(a+1)}} & k &= \sqrt{\frac{1-m}{2}} \\ cn\ u &= \sqrt{\frac{a(m+1)}{m(a+1)}} & dn\ u &= \sqrt{\frac{(m+1)(a+m)}{2m(a+1)}} \end{aligned}\right\} \quad (A4)$$

$$a = \frac{m\ cn^2u}{1+m\ sn^2u} \qquad \frac{da}{\sqrt{m^2-a^2}\sqrt{a+a^2}} = -\sqrt{\frac{2}{m}}\,du$$

The terms involving $\frac{1}{m_t-a}$ require in addition the following functions and parameters:

$$n = \frac{m(m_t+1)}{m_t-m} \text{ and } \alpha = tn^{-1}\left(\sqrt{\frac{n}{k^2}} , k'\right) \qquad (A5)$$

In the final formula for I_2, given below, the elliptic functions have been re-expressed, as far as possible, in terms of the original parameters, in order to simplify the computing procedure. The result is

[3]The symbols u and α used throughout the remainder of this appendix are the standard notation for elliptic functions and are not to be confused with the aerodynamic symbols of the text.

$$I_2 = \frac{1}{m_t\sqrt{2mm_t(1+m_t)}}\left(\left[\frac{m_t-a_t}{m_t}\left(\frac{m_t-a_t}{m_t+1}-\frac{a_t}{m}\right)-\frac{a_t(m-a_t)}{m^2}\right]F(\varphi,k)\right.$$

$$+\frac{2a_t}{m}\left[\left(1+\frac{m_t-a_t}{m_t}\right)E(\varphi,k)+\left(a_t-\frac{m_t-a_t}{m_t}\right)\sqrt{\frac{m^2-a_t^2}{2ma_t(a_t+1)}}\right]$$

$$-\frac{(1+m)(m_t-a_t)^2}{(m_t-m)^2}\left\{\left[1+\frac{m_t}{1+m_t}+\frac{m}{m_t(1+m_t)}\right]\Pi_3(n,k,\varphi)\right.$$

$$\left.\left.+\frac{2m(1+m)}{m_t-m}\frac{\partial\Pi_3(n,k,\varphi)}{\partial n}\right\}\right) \tag{A6}$$

where

$$\varphi = \sin^{-1}\sqrt{\frac{m-a_t}{m(a_t+1)}}$$

and

$$\Pi_3 = \int_0^{u(a_t)}\frac{du}{1+n\,\mathrm{sn}^2u}$$

is Legendre's form of the elliptic integral of the third kind.

In the present problem the integral

$$\int_0^{u(a_t)}\frac{du}{(1+n\,\mathrm{sn}^2u)^2}$$

also arises. Differentiation of Π_3 with respect to the parameter n gives

$$\frac{\partial\Pi_3}{\partial n} = -\int_0^{u(a_t)}\frac{\mathrm{sn}^2u\,du}{(1+n\,\mathrm{sn}^2u)^2}$$

$$= \frac{1}{n}\int_0^{u(a_t)}\frac{du}{(1+n\,\mathrm{sn}^2u)^2}-\frac{1}{n}\int_0^{u(a_t)}\frac{du}{1+n\,\mathrm{sn}^2u}$$

which may then be solved for

$$\int_0^{u(a_t)}\frac{du}{(1+n\,\mathrm{sn}^2u)^2} = \Pi_3 + n\frac{\partial\Pi_3}{\partial n}$$

This relation is the source of the term in $\partial\Pi_3/\partial n$ in equation (A6). While Π_3 and its derivative are not tabulated, they can be evaluated for individual cases by the following method:

We first note that $n > o$. The procedure to evaluate Π_3 required in that case has been outlined in some detail as case IV in reference 13. The following is an adaptation of that procedure to make use of available tabulated functions. The parameter α (equation (A5)) is introduced. Then, if u_t is the value of u when $a = a_t$,

$$\Pi_3\,(n,k,\varphi) = u_t\,\left\{1 + \frac{\operatorname{sn}(\alpha,k')\operatorname{cn}(\alpha,k')}{\operatorname{dn}(\alpha,k')}\left[iZ(i\alpha) + \frac{1}{u_t}\,\psi\right]\right\} \tag{A7}$$

with

$$iZ(i\alpha) = \frac{\pi\alpha}{2KK'} + E(\alpha,k') - \alpha\,\frac{E'}{K'} - \frac{\operatorname{dn}(\alpha,k')\operatorname{sn}(\alpha,k')}{\operatorname{cn}(\alpha,k')}$$

and

$$\psi = \tan^{-1}\frac{2q\,\sin 2a_n u_t\,\sinh 2a_n\alpha - 2q^4\sin 4a_n u_t\,\sinh 4\,a_n\alpha + \ldots}{1 - 2q\,\cos 2a_n u_t\,\cosh 2a_n\alpha + 2q^4\,\cos 4a_n u_t\,\cosh 4a_n\alpha - \ldots}$$

The elliptic functions $\operatorname{sn}(\alpha,k')$, $\operatorname{cn}(\alpha,k')$, $\operatorname{dn}(\alpha,k')$ can be computed from the value of $\operatorname{tn}(\alpha,k')$ (equation (A5)), while u_t is simply $F(\varphi,k)$. In $Z(i\alpha)$, K,K' and E,E' are the complete elliptic integrals of moduli k and k', respectively. In ψ, q is the quantity $e^{\frac{-\pi K'}{K}}$, and a_n is the asymptotic value of the sequence defined by the recurrence formulae $a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$ and $b_n = \sqrt{a_{n-1}\,b_{n-1}}$ with $a_o = 1$ and $b_o = k'$. (See reference 13.)

The derivative of Π_3 with respect to n is $\dfrac{\partial\Pi_3/\partial\alpha}{dn/d\alpha}$, which may be obtained for this case $(n > o)$ in the form

$$\frac{\partial\Pi_3}{\partial n} = \frac{\operatorname{sn}(\alpha,k')\operatorname{cn}(\alpha,k')}{2n\,\operatorname{dn}(\alpha,k')}\left(\left[\frac{\operatorname{cn}^2(\alpha,k')}{\operatorname{dn}^2(\alpha,k')} - \operatorname{sn}^2\,(\alpha,k')\right]\left\{\left[E(\alpha,k')\right.\right.\right.$$

$$\left.\left. + \frac{E}{K}\,\alpha - \alpha\right]u + \psi\right\} - \frac{\operatorname{sn}(\alpha,k')\operatorname{cn}(\alpha,k')}{\operatorname{dn}(\alpha,k')}\left\{\left[1 + \operatorname{dn}^2(\alpha,k')\right]u\right.$$

$$\left.\left. - E(\operatorname{am} u,k) - \frac{n\,\operatorname{sn} u\,\operatorname{cn} u\,\operatorname{dn} u}{1 + n\,\operatorname{sn} u}\right\}\right)$$

Substitution of the limits and the expressions in equation (A4), gives

$$\frac{\partial \Pi_3(n,\varphi,k)}{\partial n} = \frac{\mathrm{sn}(\alpha,k')\mathrm{cn}(\alpha,k')}{2n\ \mathrm{dn}\ (\alpha,k')}\left(\left[\frac{\mathrm{cn}^2(\alpha,k')}{\mathrm{dn}^2(\alpha,k')} - \mathrm{sn}^2(\alpha,k')\right]\left\{\left[E(\alpha,k')\right.\right.\right.$$

$$\left.\left. + \frac{E}{K}\alpha - \alpha\right]F(\varphi,k) + \Psi\right\}$$

$$- \frac{\mathrm{sn}(\alpha,k')\mathrm{cn}(\alpha,k')}{\mathrm{dn}(\alpha,k')}\left\{\left[1 + \mathrm{dn}^2(\alpha,k')\right]F(\varphi,k)\right.$$

$$\left.\left. - E(\varphi,k) - \frac{m_t+1}{m_t-a_t}\sqrt{\frac{a_t(m^2-a_t^2)}{2m(a_t+1)}}\right\}\right)$$

REFERENCES

1. von Kármán, Theodore: Supersonic Aerodynamics – Principles and Applications. Jour. Aero. Sci., vol. 14, no. 7, July 1947.

2. Evvard, John C.: Distribution of Wave Drag and Lift in the Vicinity of Wing Tips at Supersonic Speeds. NACA TN No. 1382, 1947.

3. Heaslet, Max. A., Lomax, Harvard, and Jones, Arthur L.: Volterra's Solution of the Wave Equation as Applied to Three-Dimensional Supersonic Airfoil Problems. NACA TN No. 1412, 1947.

4. Busemann, Adolf: Infinitesimal Conical Supersonic Flow. NACA TM No. 1100, 1947.

5. Puckett, Allen E.: Supersonic Wave Drag of Thin Airfoils. Jour. Aero. Sci., vol. 13, no. 9, Sept. 1946, pp. 475–484.

6. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quart. of Applied Math., vol. IV, no. 3, Oct. 1946. pp. 246–254.

7. Heaslet, Max. A., and Lomax, Harvard: The Use of Source-Sink and Doublet Distributions Extended to the Solution of Arbitrary Boundary Value Problems in Supersonic Flow. NACA TN No. 1515, 1947.

8. Legendre, Adrien Marie: Tables of the Complete and Incomplete Elliptic Integrals. Biometrika Office, University College, Cambridge University Press, Cambridge, Eng. 1934.

9. Dwight, Herbert Bristol: Tables of Integrals and Other Mathematical Data. The Macmillan Company, New York, 1934, p. 155.

10. Hayes, Wallace D., Browne, S. H., and Lew, R. J.: Linearized Theory of Conical Supersonic Flow with Application to Triangular Wings. North American Report NA-46-818., Sept. 30, 1946.

11. Jones, Robert T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds Below and Above the Speed of Sound. NACA TN No. 1032, 1946.

12. Jones, Robert T.: Estimated Lift-Drag Ratios at Supersonic Speed. NACA TN No. 1350, 1947.

13. King, Louis V.: On the Direct Numerical Calculation of Elliptic Functions and Integrals. Cambridge University Press, 1924.

TABLE I – CALCULATED LIFT

Source of lift	Method of calculating	Tapered wing		Untapered wing	
		L/qα	% total	L/qα	% total
Basic solution	Equations (11) and (12)	84.58 sq ft	105.1	151.58 sq ft	130.0
Tip effect	Equations (43) and (44)	–1.32	–1.6	–22.30	–19.1
Symmetrical wake correction	Equation (55)	–2.32	–2.9	–15.90	–13.6
Oblique elements in wake	Equations (57) and (58)	–.50	–.6	–1.24	–1.1
Second correction at trailing edge	Equation (71)	+.02	+.025	+2.66	+2.3
Second correction at tip	Equation (73)	+.01 +.002	+.012 +.003	+1.70 +.14	+1.4 +.1
Totals		80.47	100.0	116.64	100.0
$\partial C_L/\partial\alpha = L/q\alpha S$		3.09		2.01	
$\partial C_L/\partial\alpha$, per degree		.054		.035	

TABLE II – CALCULATED PITCHING MOMENT

Source of moment	Tapered wing		Untapered wing		Modified untapered wing	
	M/qα	% total	M/qα	% total	M/qα	% total
Basic	–495.5 cu ft	108.22	–1169.0 cu ft	154.7	–701.8 cu ft	105.0
Tip effect	+13.5	–2.95	+279.2	–37.0	0	0
Symmetrical wake correction	+19.4	–4.24	+179.8	–23.8	+33.4	–5.0
Oblique elements in wake	+4.7	–1.03	+16.0	–2.1	– – –	– – –
Second correction at trailing edge	– – –	– – –	–38.3	+5.1	0	0
Second correction at tip	– – –	– – –	–23.4	+3.1	0	0
Totals	–457.9	100.0	–755.7	100.0	–668.4	100.0
Moment arm of lift / Moment arm of area	1.08		0.83		1.01	

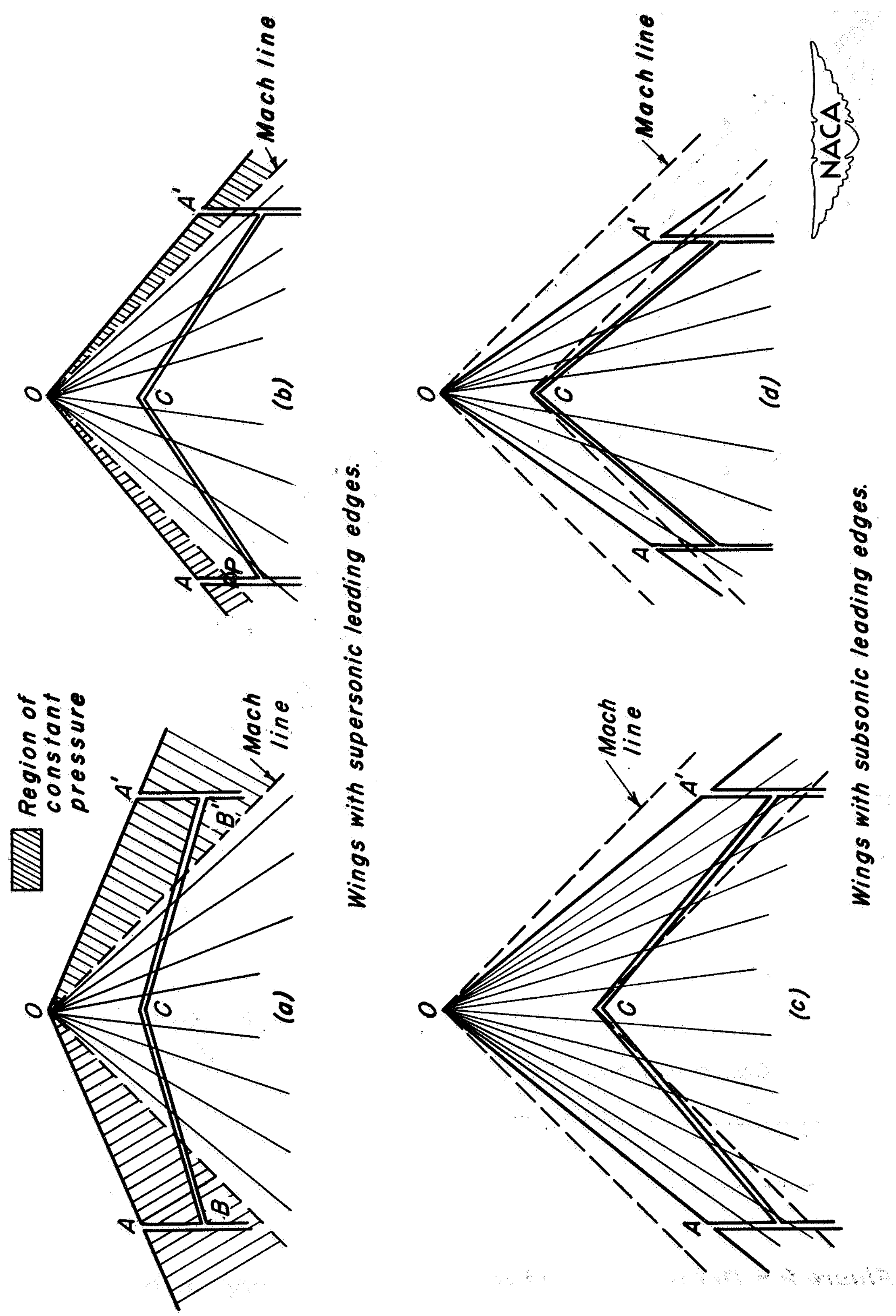


Figure 1.- Conical flow regions on wings with various degrees of sweep relative to the Mach lines.

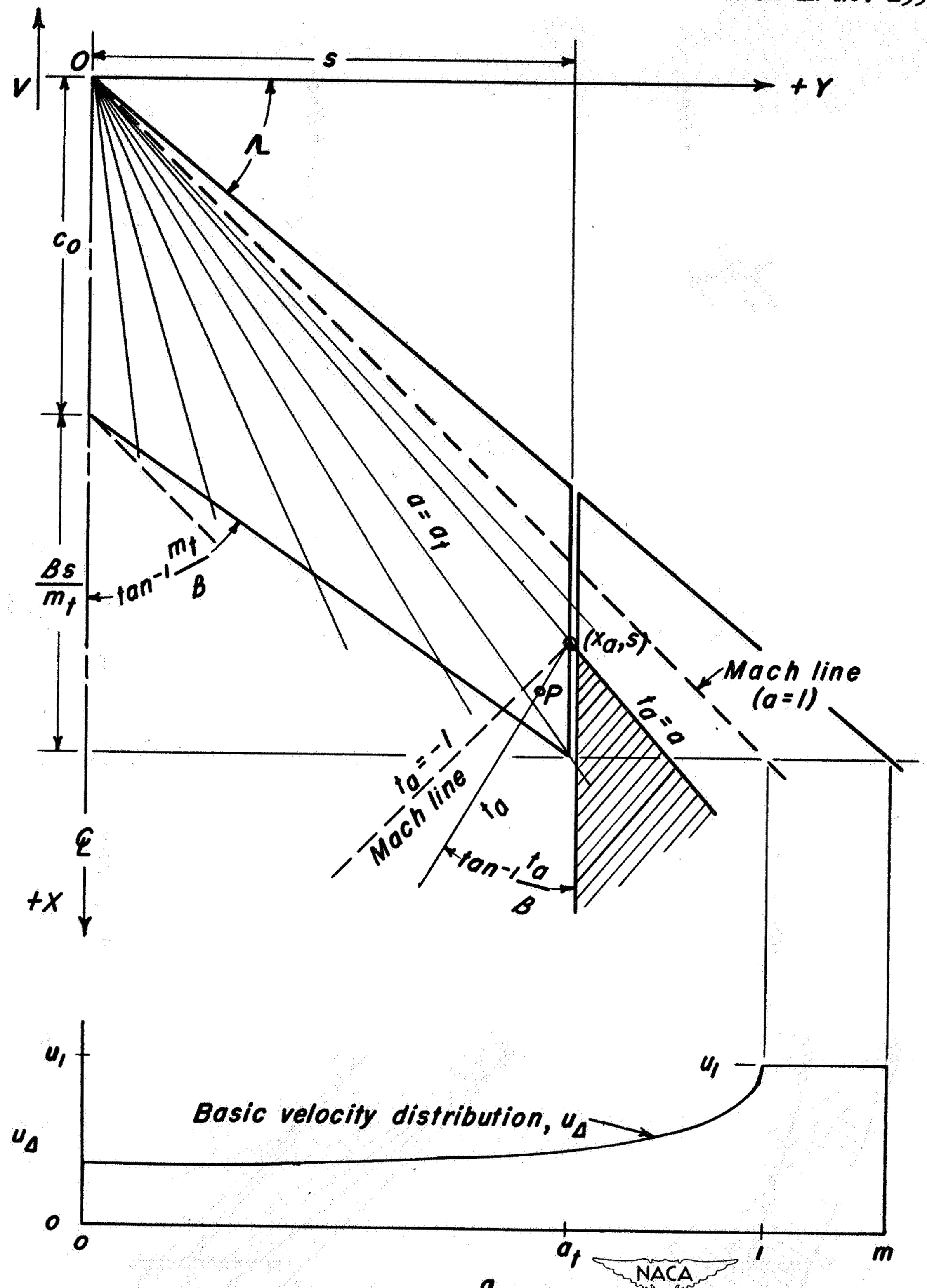


Figure 2.– Detail of tip of low aspect ratio wing with supersonic leading edge, showing axes and symbols used in formulas, and basic velocity distribution.

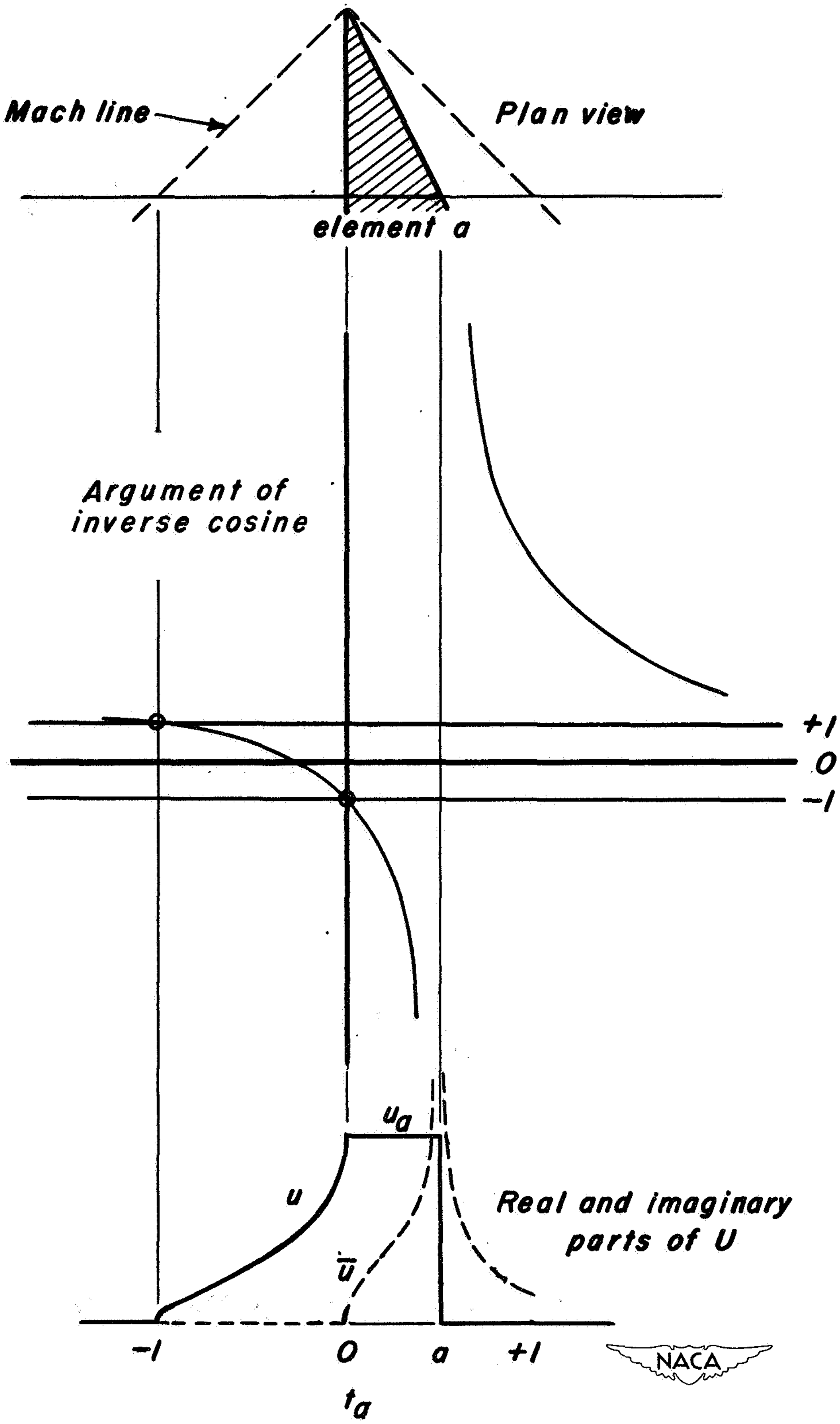


Figure 3.– Elementary solution for tip.

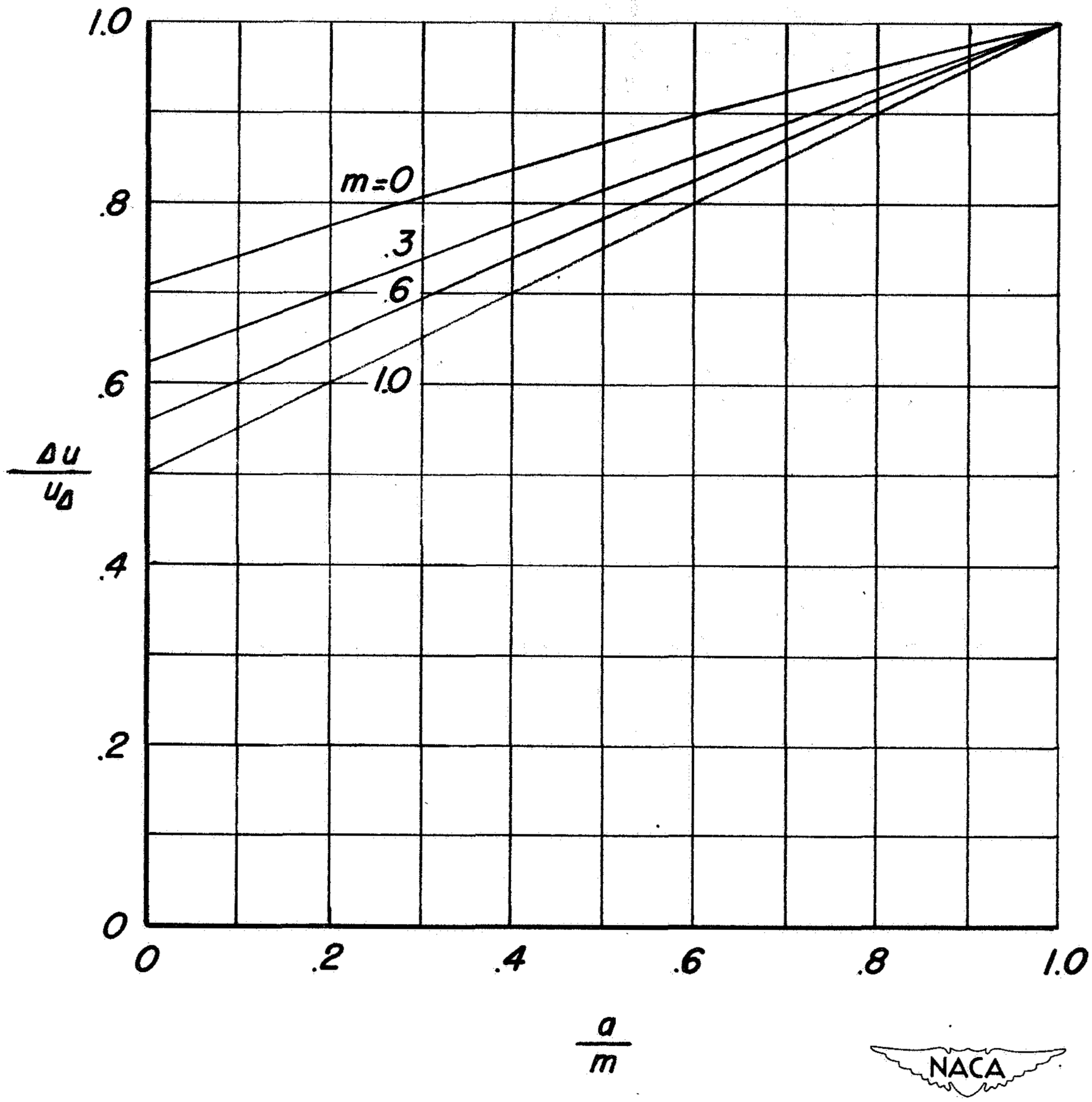


Figure 4.- Percent dropoff in lift along Mach line from tip.

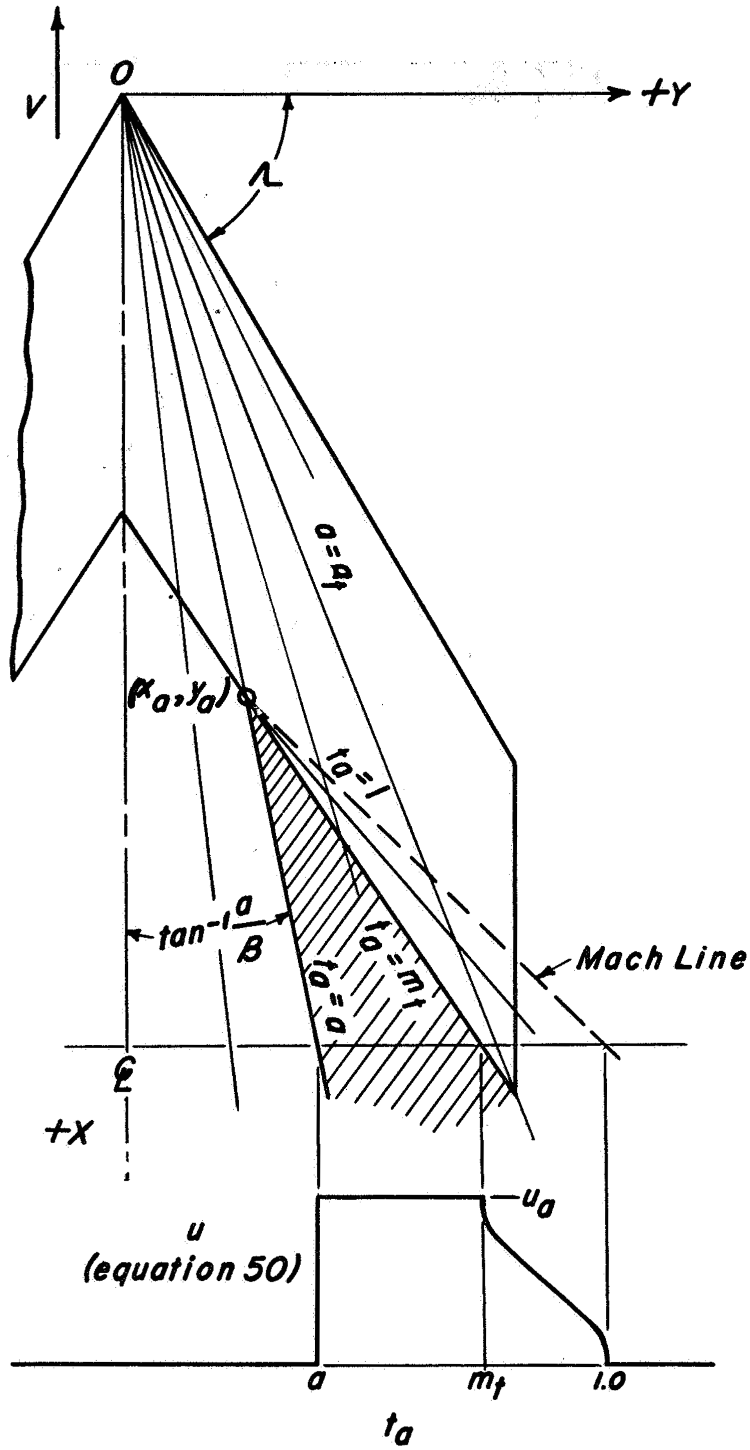


Figure 5.– Detail of trailing edge of highly swept wing showing oblique constant-lift element and induced velocity distribution.

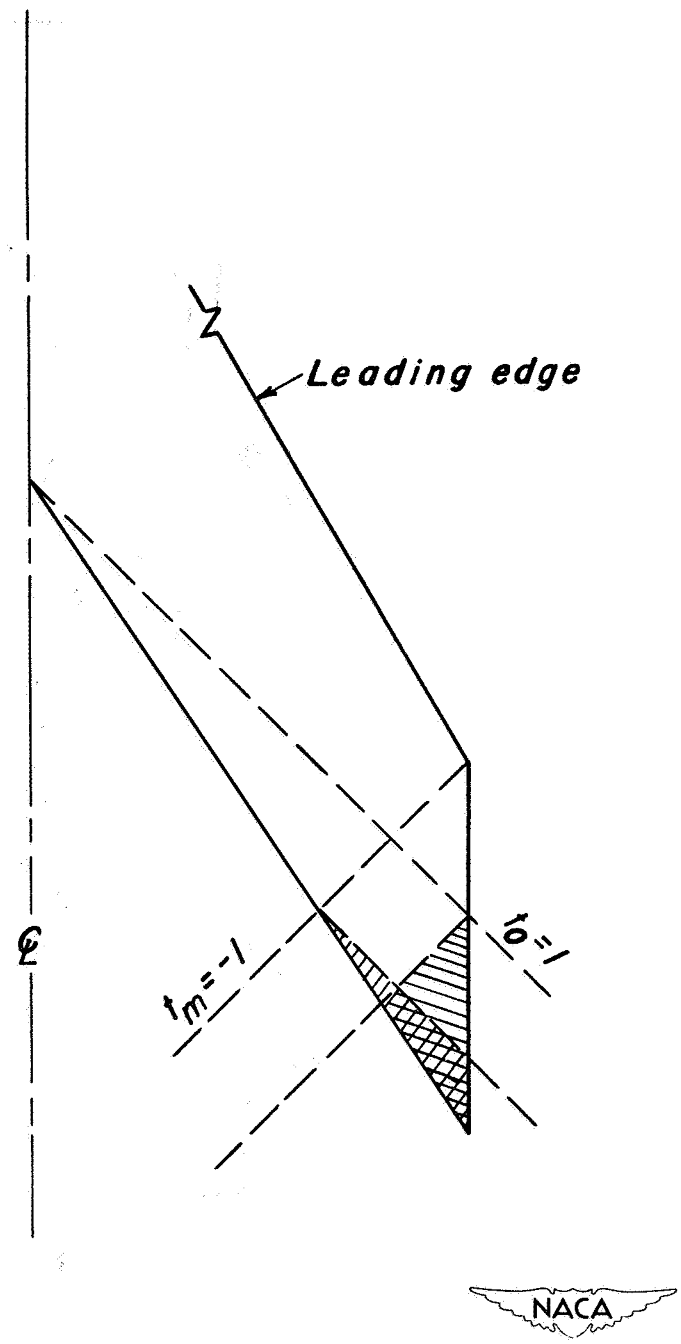


Figure 6.– Regions of tip and trailing edge interaction.

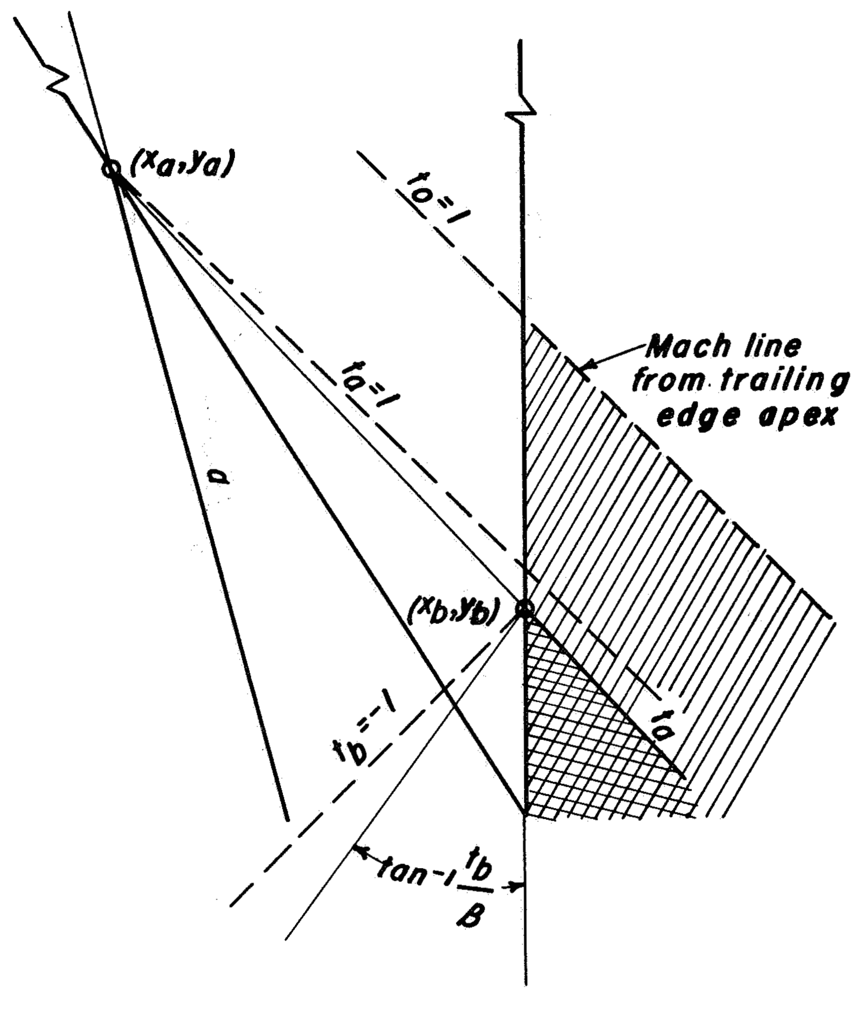


Figure 7.- Second step of successive approximations. Correction for pressures at tip induced by trailing-edge solution. Shaded area denotes extraneous pressure field.

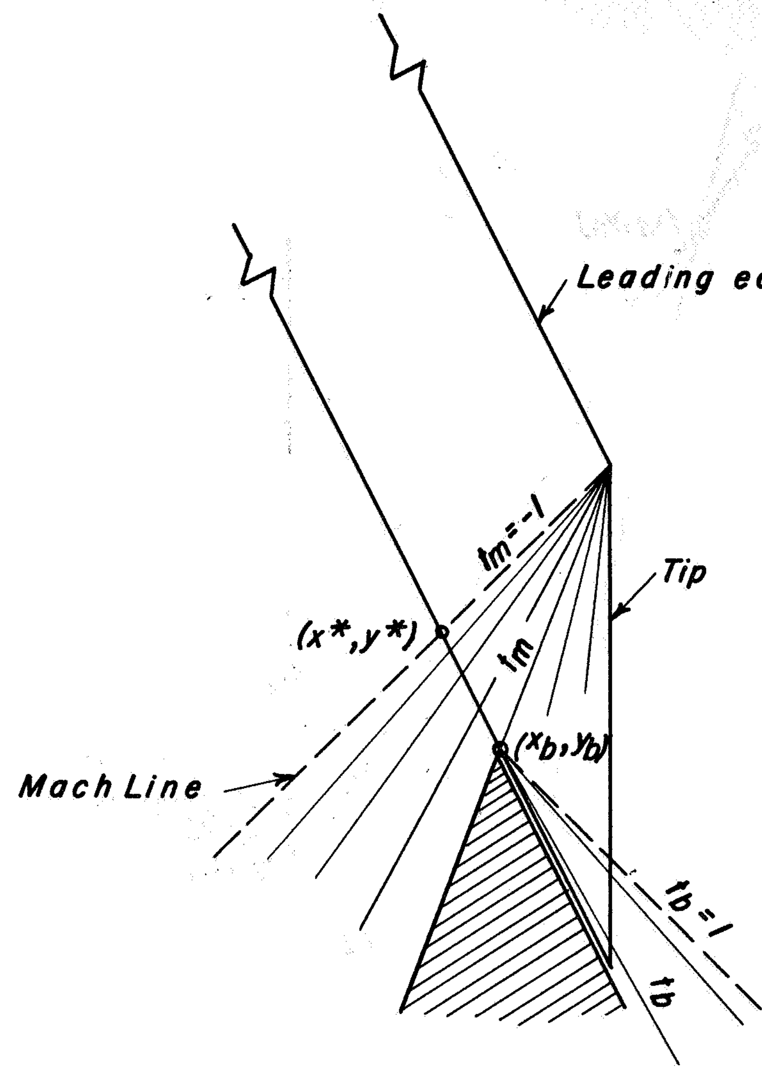


Figure 8.— Approximate cancellation of extraneous pressures behind trailing edge by single conical field from the leading edge tip.

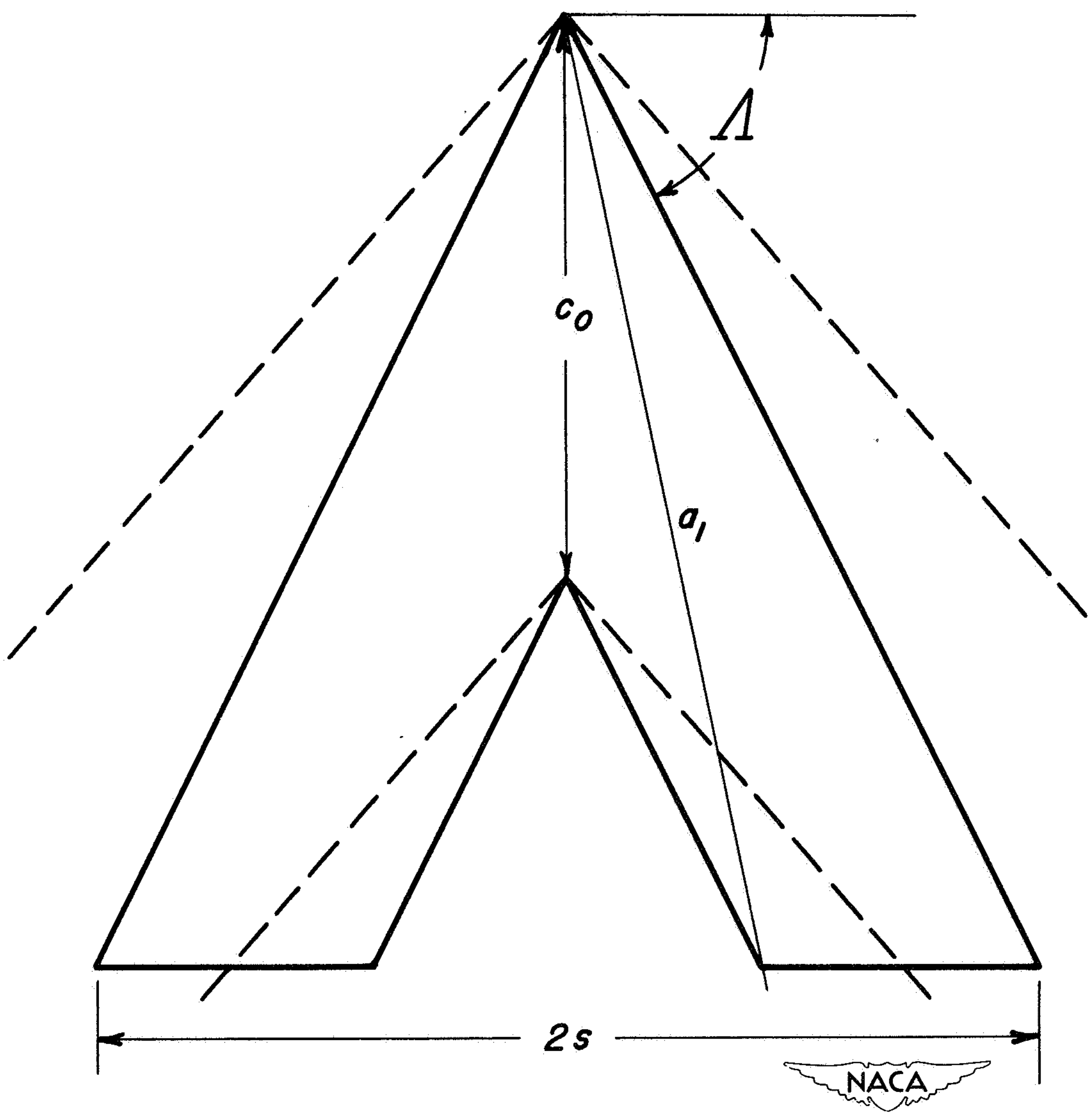


Figure 9.— Wing with "cross-stream tips", showing ray a_1 through the inboard end of the tip.

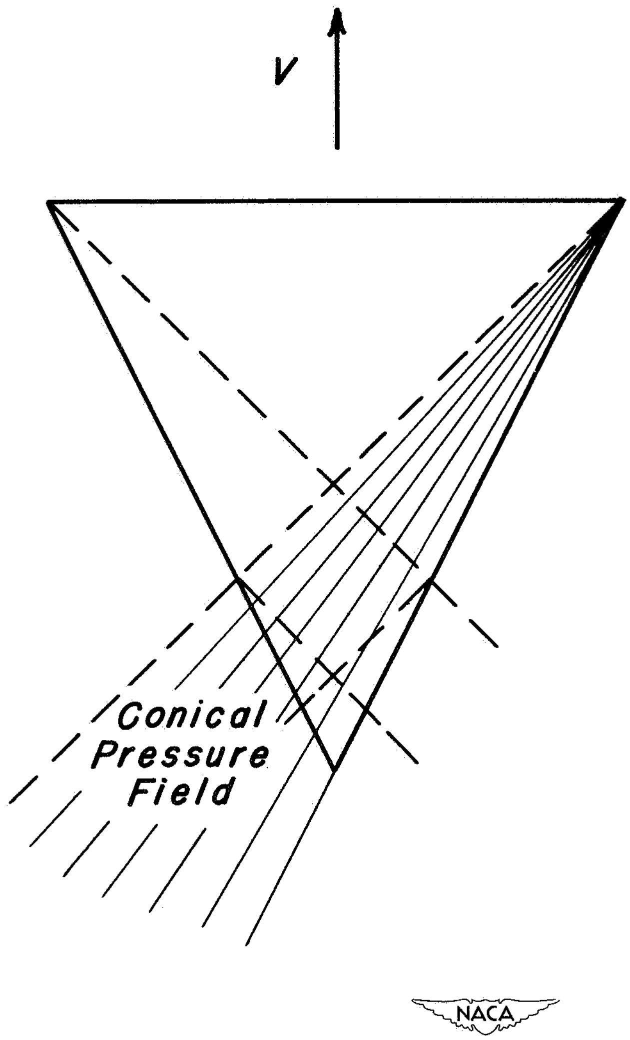


Figure 10.— Reverse delta wing.

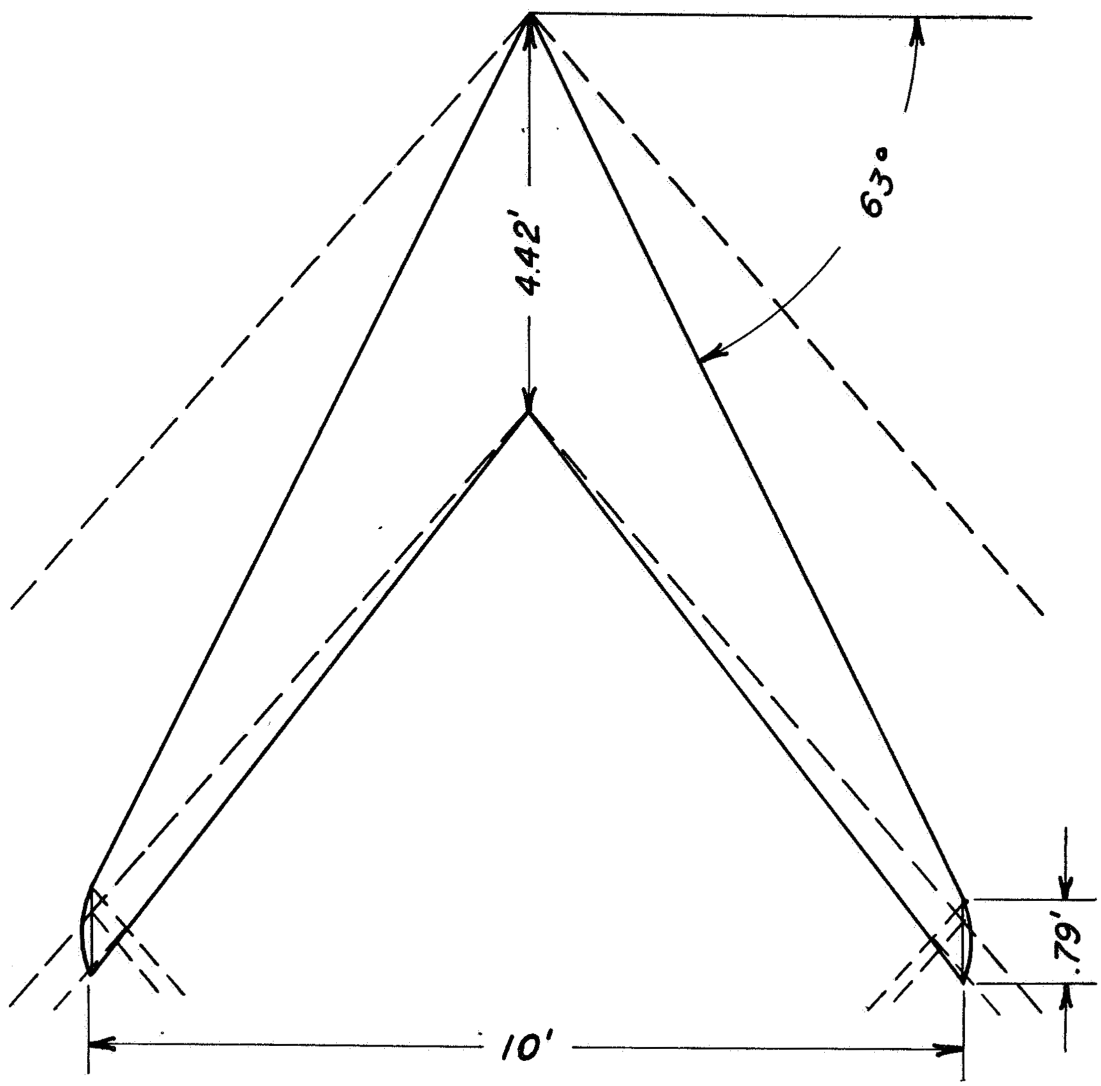


Figure 11.– Tapered wing used in calculations showing primary and secondary Mach lines at M=1.5.

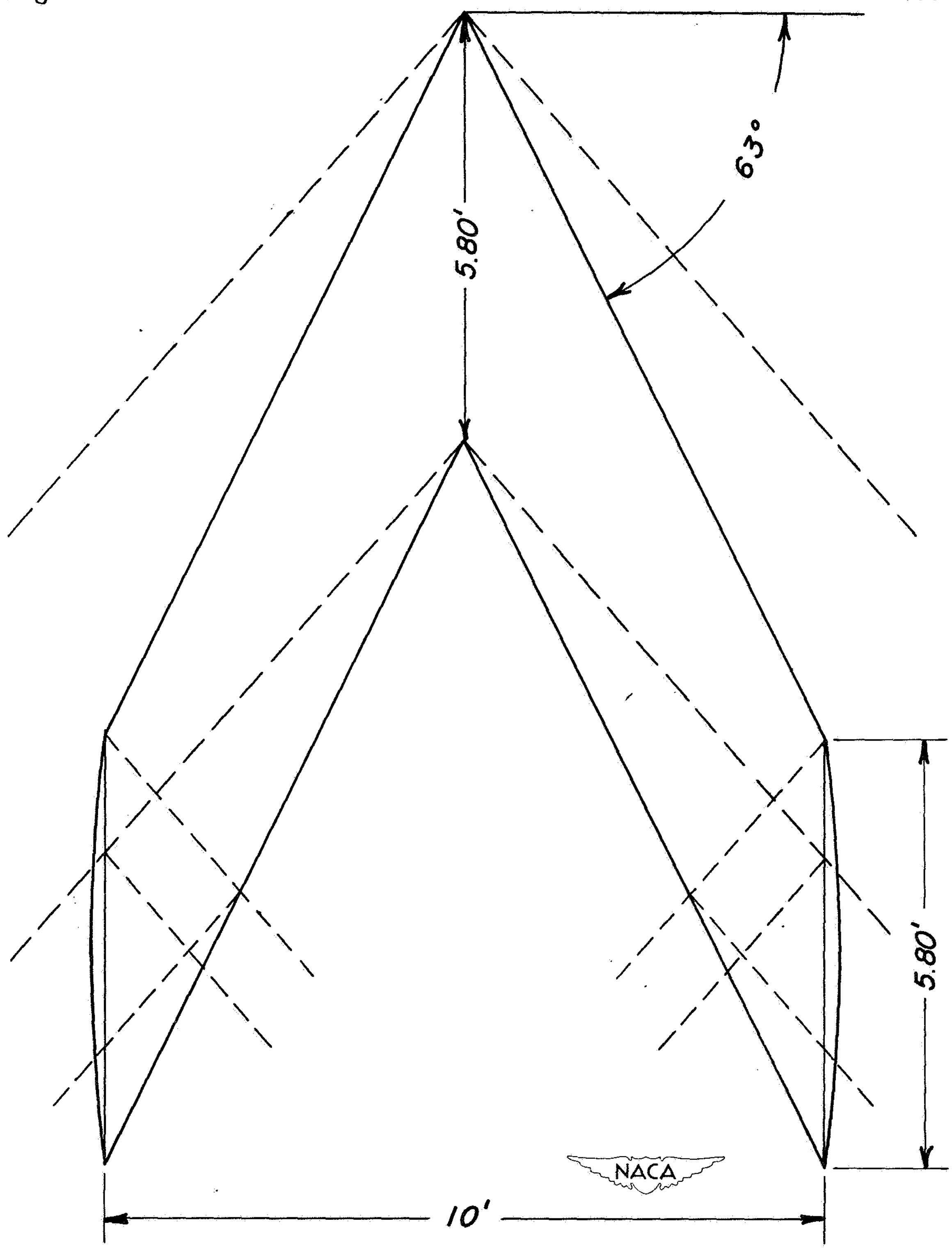


Figure 12.– Untapered wing for which pressure distribution was calculated, showing primary and secondary Mach lines at M=1.5.

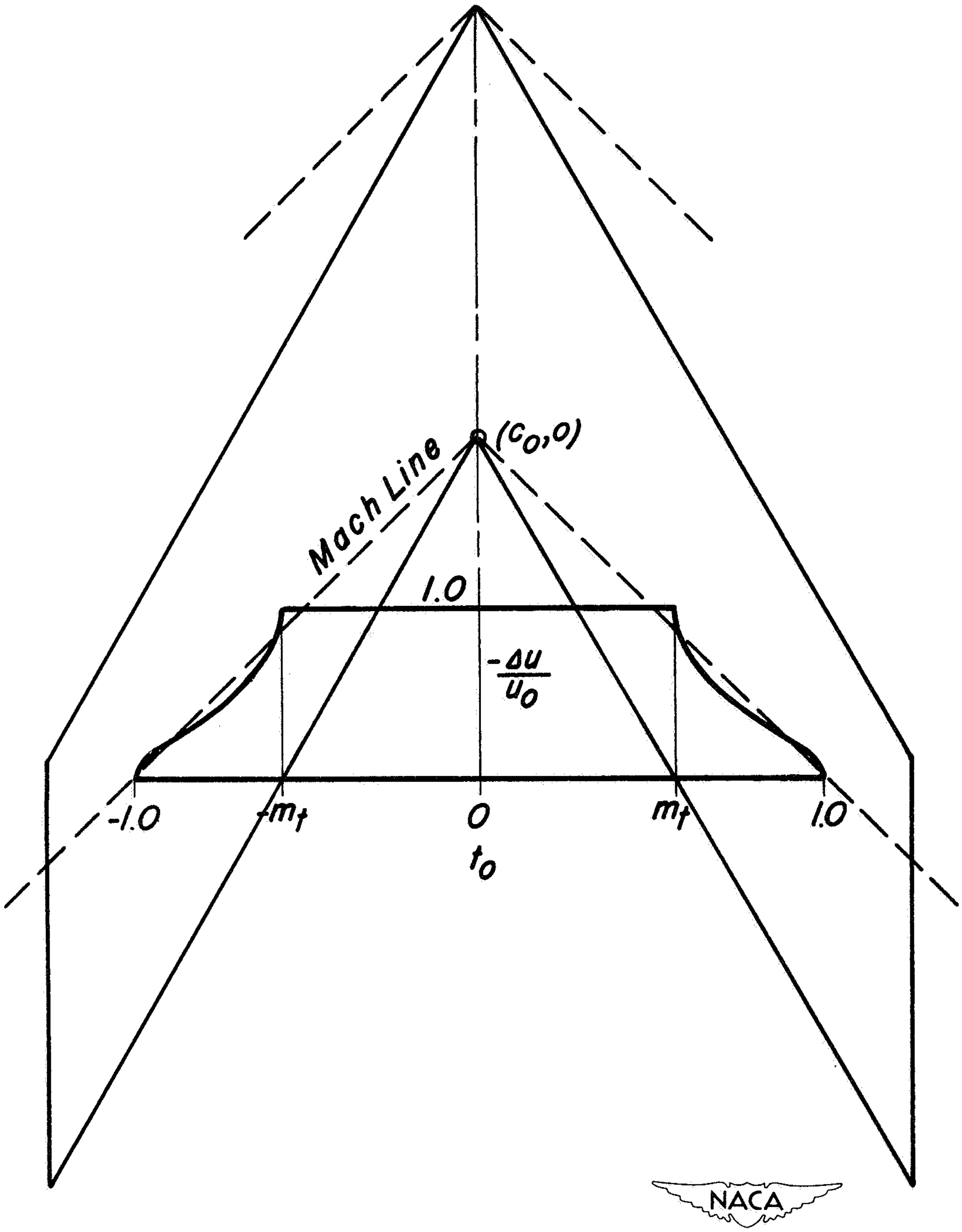


Figure 13.– Incremental velocity distribution due to removal of constant-lift symmetrical sector in wake (shown positive).

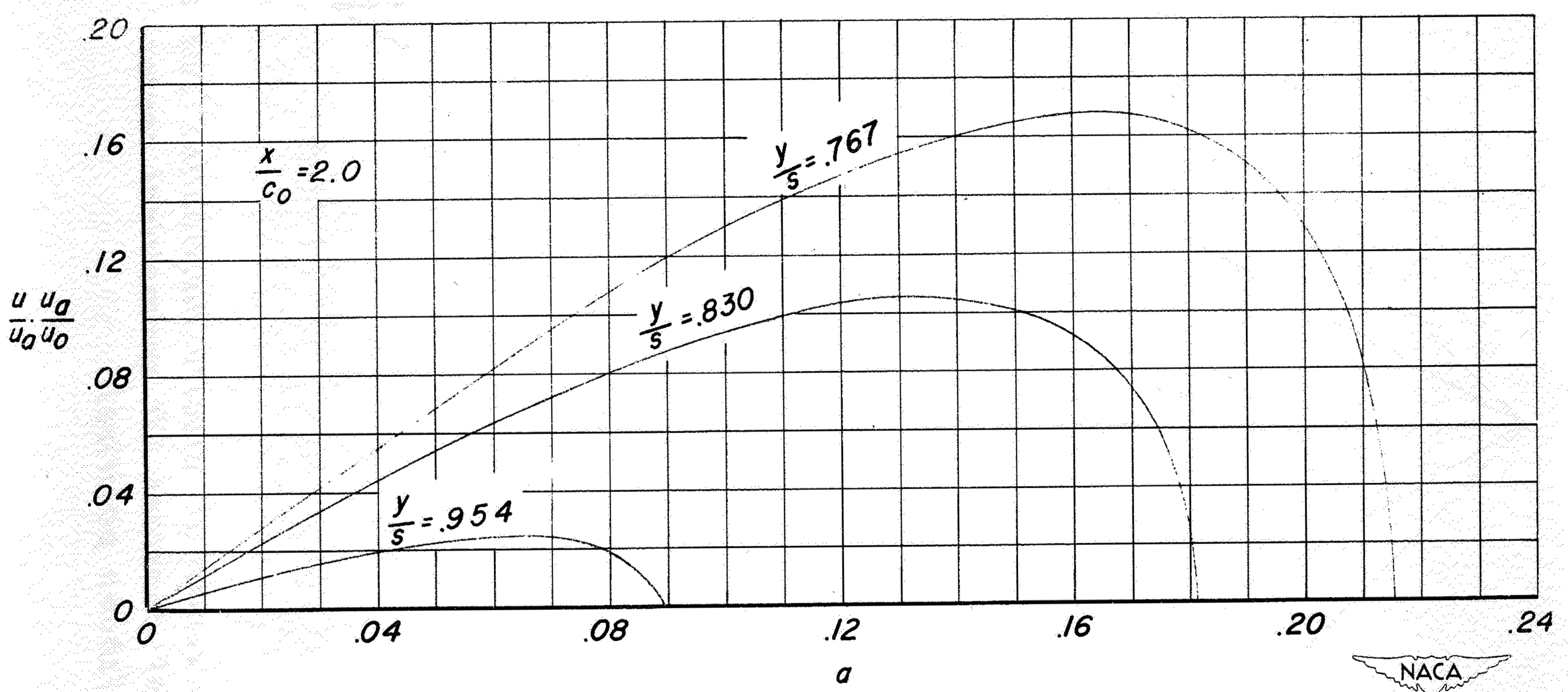


Figure 14.– Typical plots of equation (50) against the parameter a, used to determine the pressure at points (x,y). The plots shown are for points located along the line $x=2c_o$.

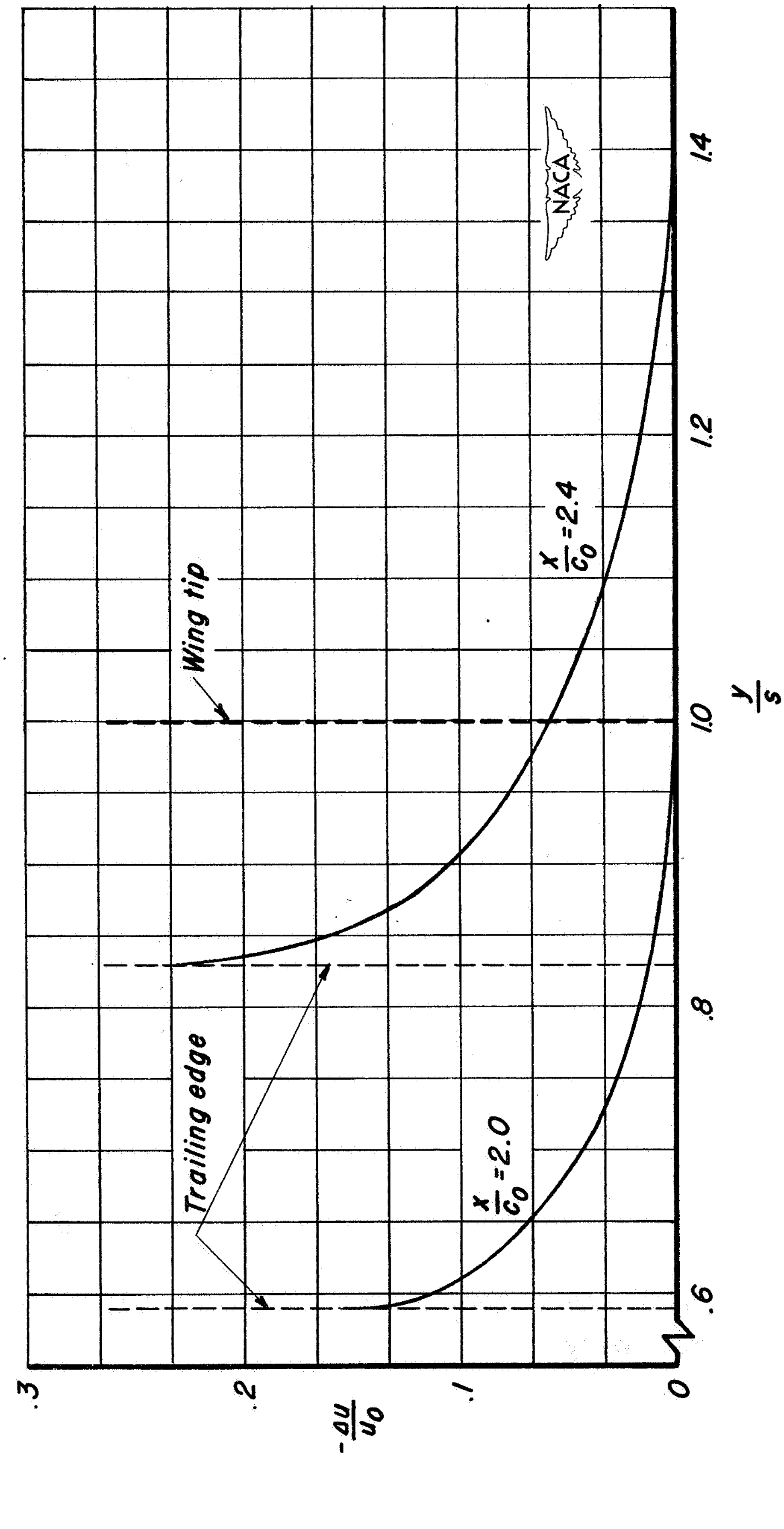


Figure 15.- Two typical spanwise incremental velocity distributions, representing modifications to the basic solution within the Mach cone from the trailing edge apex (contribution of oblique solutions only).

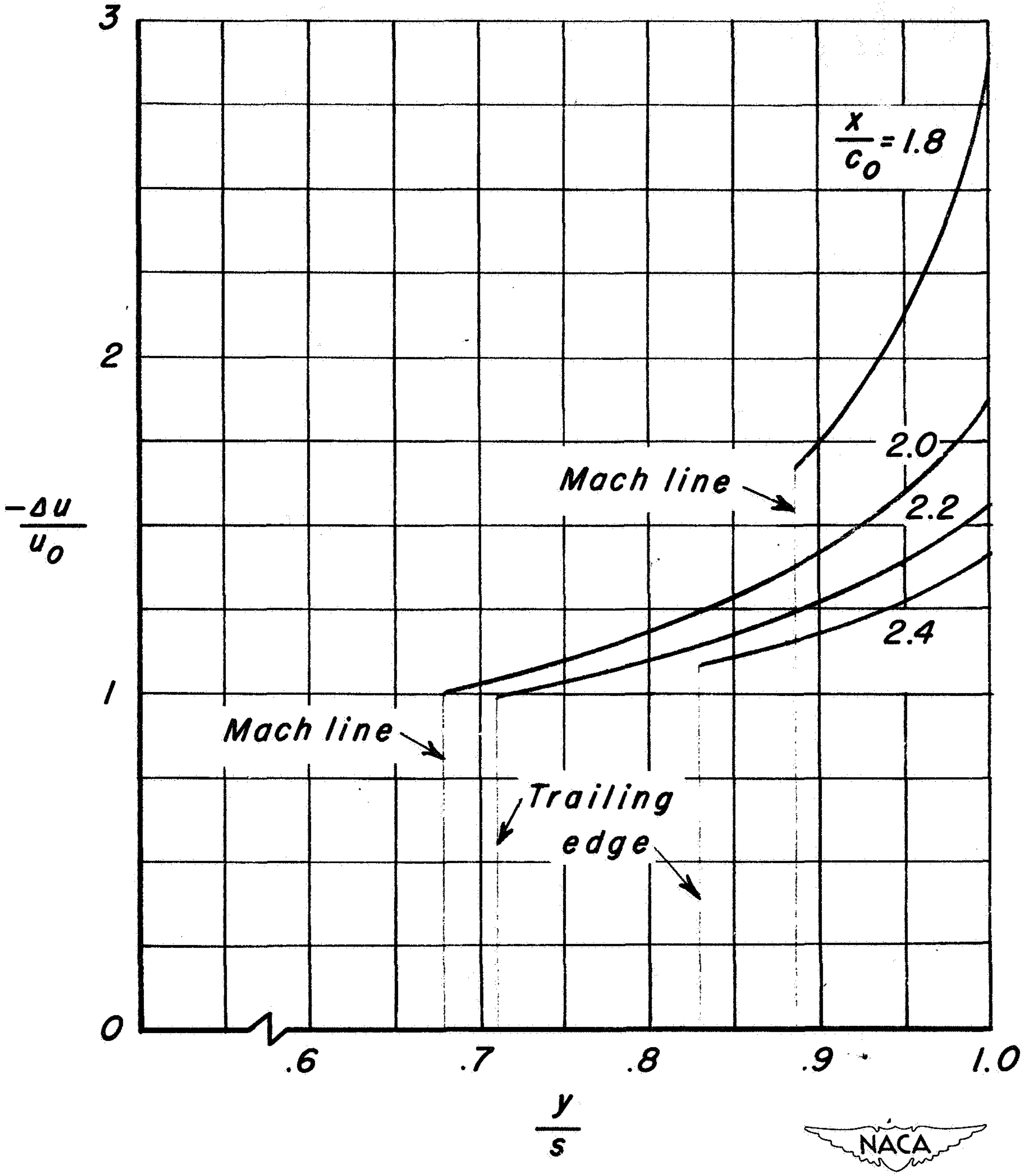


Figure 16.— Spanwise distribution of incremental velocity at various chordwise stations within the Mach cone from the tip of the untapered wing.

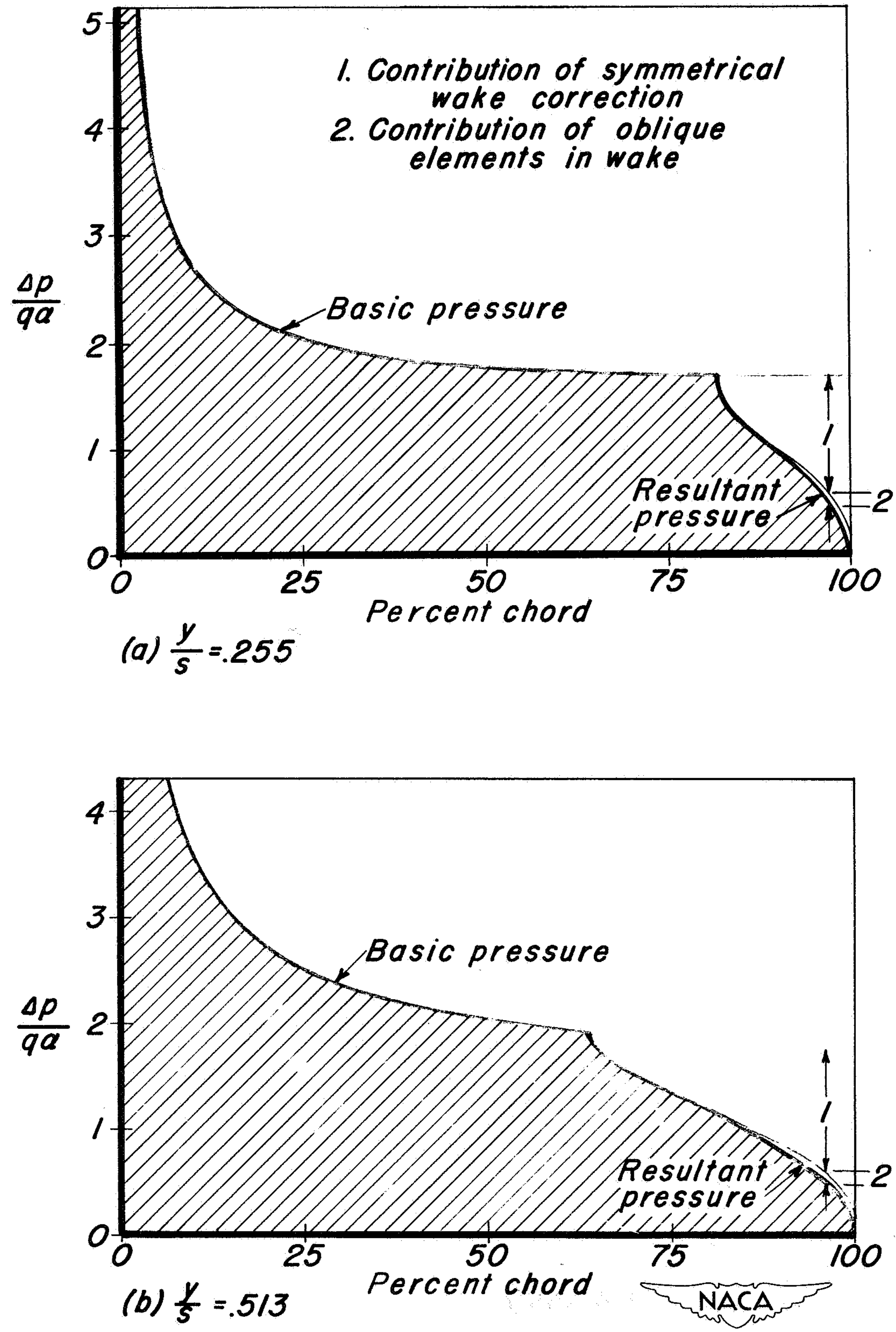
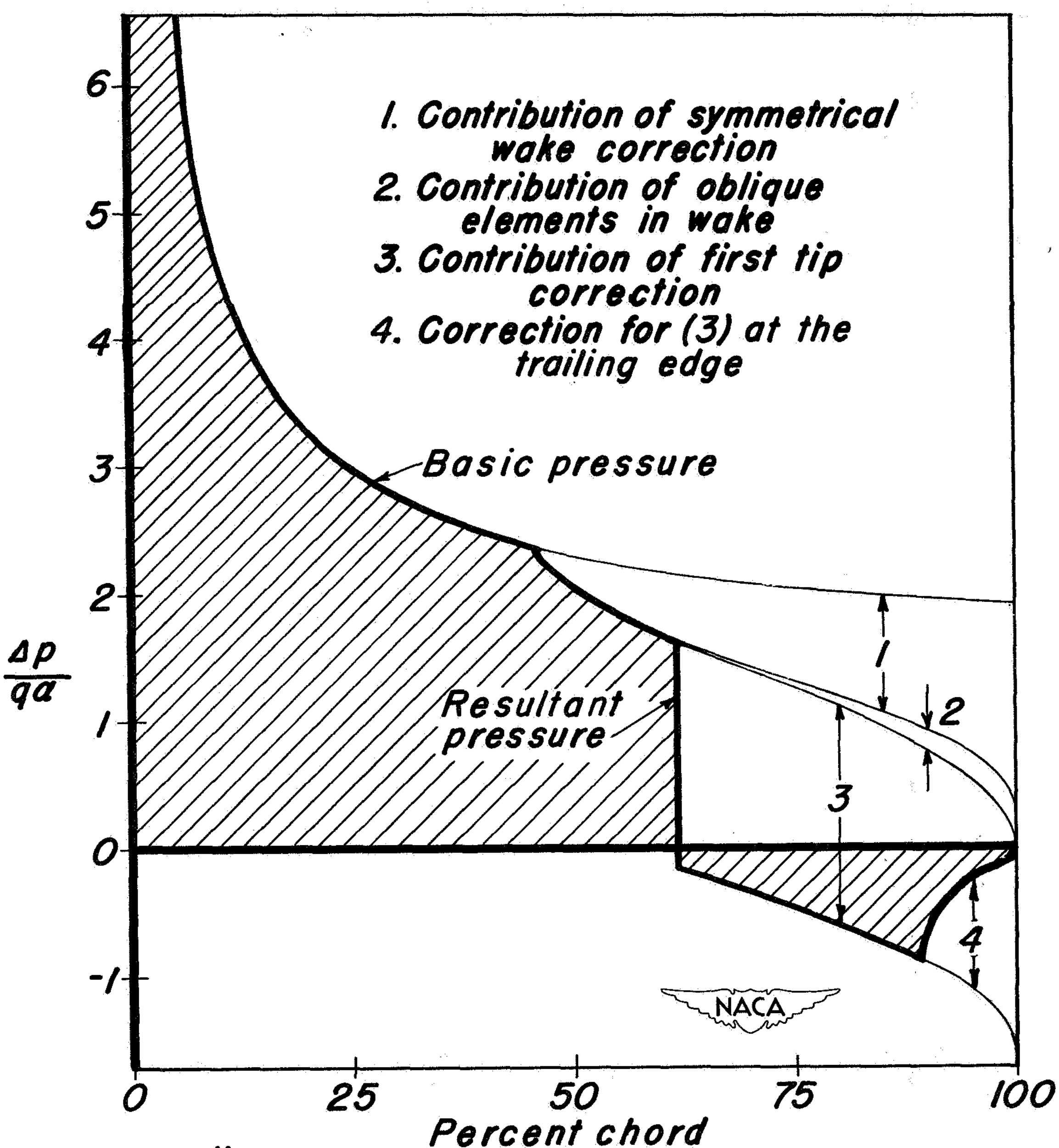
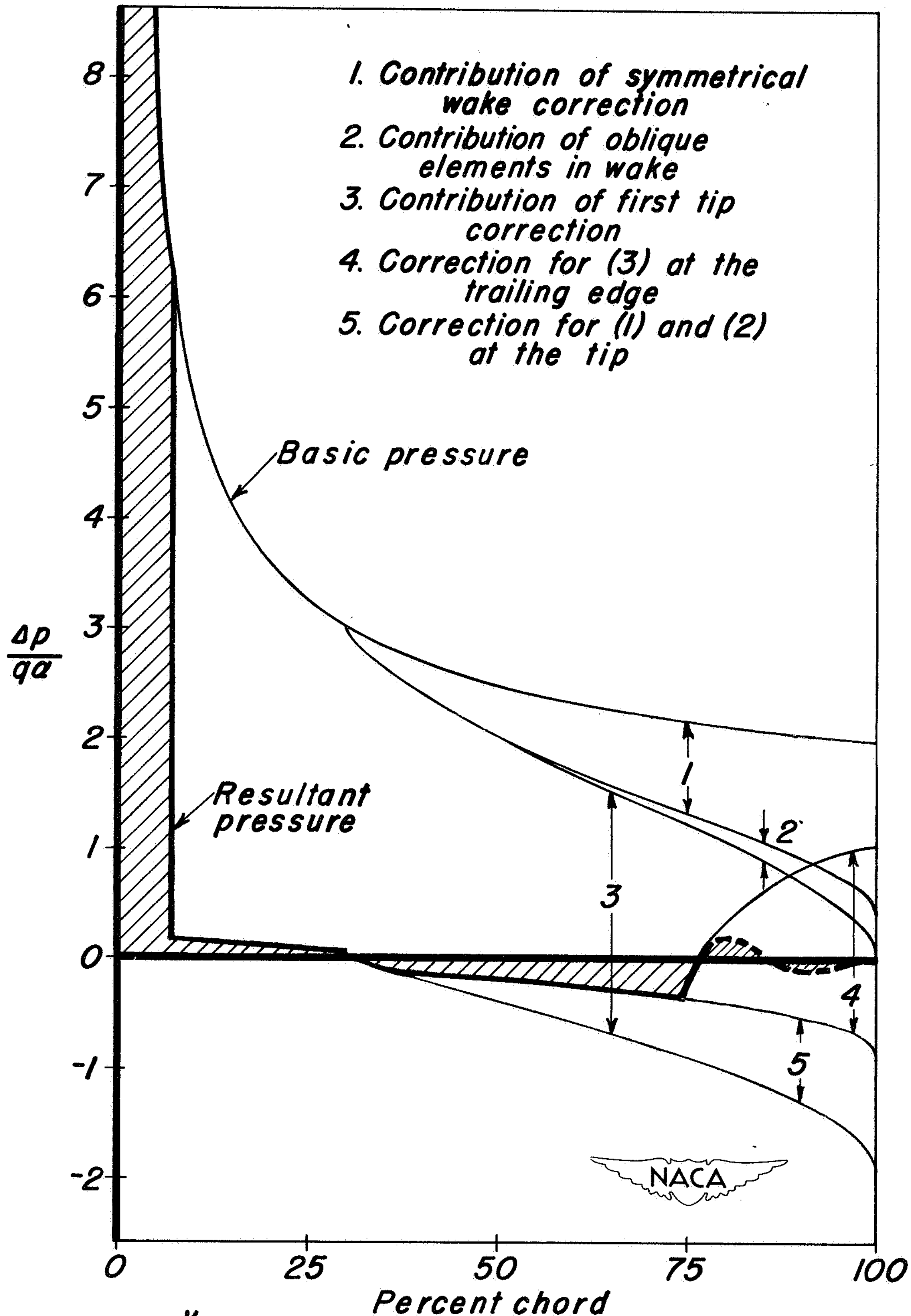


Figure 17.– Chordwise pressure distributions calculated for the untapered wing.



(c) $\frac{y}{s}$ =.768

Figure 17.– (continued)



(d) $\frac{y}{s} = .975$

Figure 17.– (concluded)

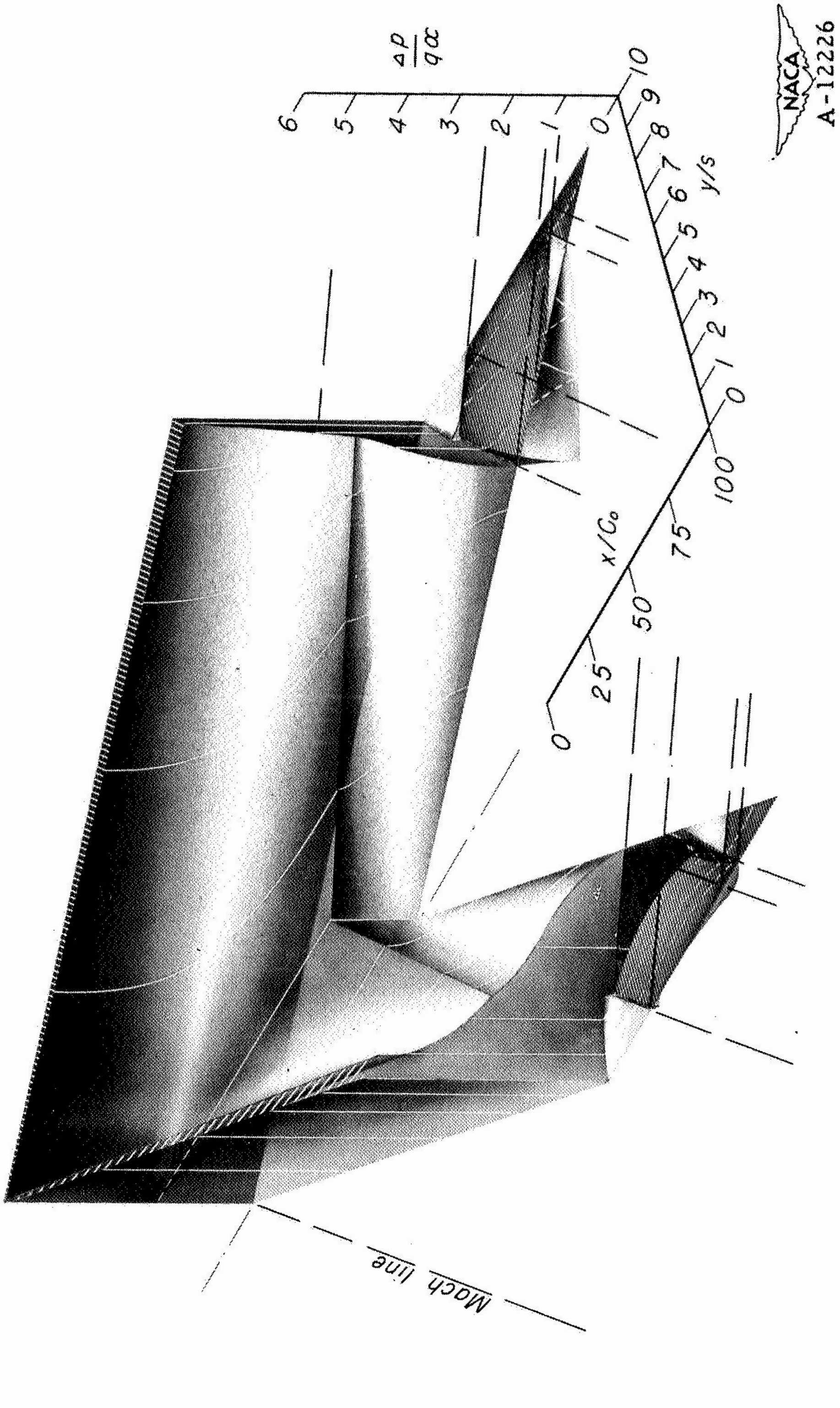


Figure 18.— Three-dimensional drawing of calculated pressure distribution over untapered 63° swept wing at M = 1.5.