The Hydrostatic Equations

Ralph L. Carmichael

January 23, 2003

Geopotential & Geometric Altitude

The standard atmosphere is defined in terms of geopotential altitude. The idea behind this concept is that a small change in geopotential altitude will make the same change in gravitational potential energy as the geometric altitude at sea level. Mathematically, this is expressed as g dZ = G dH where H stands for geopotential altitude and Z stands for geometric altitude, g is the acceleration of gravity and G is the value of g at sea level. The value of g varies with altitude and is shown in elementary physics texts to vary as

$$\frac{g}{G} = \left(\frac{E}{Z+E}\right)^2 \tag{1}$$

where E is the radius of the earth. So,

$$dH = \frac{g}{G}dZ = \left(\frac{E}{Z+E}\right)^2 dZ \tag{2}$$

and integrating yields

$$\int_0^H dH = \int_0^Z \left(\frac{E}{Z+E}\right)^2 dZ \tag{3}$$

$$H = \frac{EZ}{E+Z} \tag{4}$$

$$Z = \frac{EH}{E-H} \tag{5}$$

While Z and H are virtually identical at low altitudes, you can calculate that Z = 86 km corresponds to H=84.852 km. (Use 6356 km for the radius of the earth). At this altitude, g is 0.9735 times the value at sea level. If you don't like the definition of H as a differential, you can regard H=EZ/(E+Z) as the definition of H and then derive dH/dZ=g/G.

The Perfect Gas Law

The equation of state of a perfect gas is

$$\rho = \frac{MP}{RT} \tag{6}$$

where P is the atmospheric pressure, R is the universal gas constant, T is the absolute temperature and M is the mean molecular weight of air. M is assumed constant (=28.9644) up to 86 km where dissociation and diffusive separation become significant. R is 8.31432 joules $K^{-1}mol^{-1}$.

The Hydrostatic Equations

The fundamental equation is

$$dP = -\rho g dZ = -\rho G dH \tag{7}$$

and using the perfect gas law gives

$$dP = -\frac{MP}{RT} G dH \tag{8}$$

This equation leads directly to the calculation of pressure in the standard atmosphere. Within an atmospheric layer, the temperature T is a linear function of the geopotential altitude H.

$$T = T_b + L(H - H_b) \tag{9}$$

where L is the constant gradient of temperature and T_b and H_b are the temperature and geopotential altitude at the base of the layer. The hydrostatic equation then becomes

$$dP = -\frac{MG}{R} \frac{P}{(T_b + L(H - H_b))} dH \tag{10}$$

and the pressure at any value of H within this layer is found by integration of this equation

$$\int_{P_b}^{P} \frac{dP}{P} = -\int_{H_b}^{H} \frac{MG}{R(T_b + L(H - H_b))} dH \tag{11}$$

The right hand integral takes different forms, depending upon whether L is zero or not. When L=0, the integral is

$$\ln\left(\frac{P}{P_b}\right) = -\frac{GM}{T_b R}(H - H_b) \tag{12}$$

and when L is not zero, the integral is

$$\ln\left(\frac{P}{P_b}\right) = -\frac{GM}{RL} \ln\left(\frac{T_b + L(H - H_b)}{T_b}\right) \tag{13}$$

Writing these equations in exponential form, when L=0

$$\frac{P}{P_b} = \exp\left(-\frac{GM(H - H_b)}{RT_b}\right) \tag{14}$$

and

$$\frac{P}{P_b} = \left(\frac{T_b + L(H - H_b)}{T_b}\right)^{-\frac{g_0 M}{RL}} \tag{15}$$

when L is not zero.

You can see now why geopotential altitude is used for the definition of the standard atmosphere. If Z were used, then g would appear in the equations instead of G and the variation of g with altitude would have to be included in the integration, making a rather complicated equation.